Section 19
or
What is a *Differential*?

What *is* a differential?

NB : Most of this is for the Gauss 1799 paper,
but the last example is useful for the Potential paper.

Gauss uses something new in parts II and III of his proof in this section: the differential. It appears first as $\frac{dT}{d\phi}$, which is read: "dee T dee Phi." Lazare Carnot's *Reflections on the Metaphysics of the Infinitesimal Calculus* is a good work on the pedagogy behind the use and nature of infinitesimals (available from http://www.wlym.com/~oakland). In brief, $\frac{dT}{d\phi}$ means "how does T change in regards to a change in ϕ." The real difference between the real calculus and what you learned in school is that a differential is a physical principle which is united, by Dirichlet's principle, to the integral trajectory that it creates. (See RAD 59 and "The Real Calculus Vs What You Learned in School," and RAD 58 on Dirichlet) We will use a couple of examples to get at what a differential is, and then come back to Gauss's paper to look at why, with

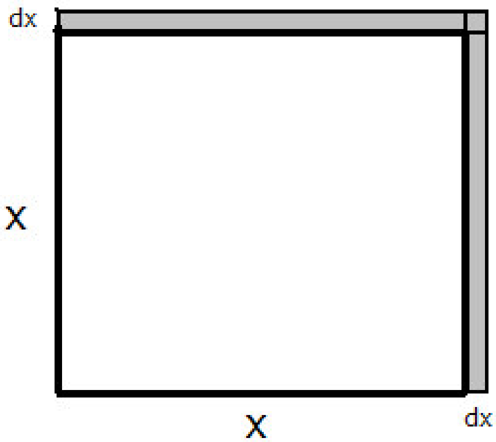
$$T = R^m \sin m\,\phi + A\,R^{m-1} \sin(m-1)\,\phi + B\,R^{m-2} \sin(m-2)\,\phi + \text{etc.}$$ (upper-case R in this section)

you get

$$\frac{dT}{d\phi} = m\,R^{m-2}(R \cos m\,\phi + \frac{m-1}{m}\,A \cos(m-1)\,\phi + \text{etc.})$$

First, examples

If you have taken calculus in school, you no doubt learned a number of rules, such as $\frac{d\,x^2}{d\,x} = 2\,x$ and generally, that $\frac{d\,x^n}{d\,x} = n\,x^{n-1}$. Let's look at the examples of squaring and cubing. (It would be helpful to play with the examples in the Fall 2003 *21st Century* of square and cube differences.)



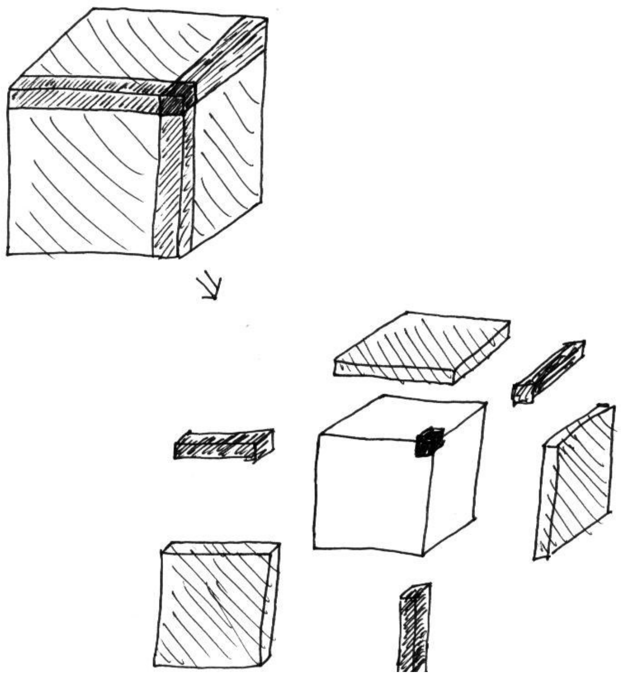
Here you can see a square that has grown a tiny amount on each side, dx. Let's look at the relationship between that increase in area and the increase in the length of the side of the square. The *21st Century* article deals with these differences for actual numbers, but here we will look at the infinitesimal. The **L**-shaped area added to the square is the d x^2 (d meaning difference), and it is composed of two rectangles, each $x \cdot d\,x$ in area, and a small square, $d\,x \cdot d\,x$. Put together, we have $2\,x\,d\,x + (d\,x)^2$. Now, if $d\,x$ is infinitely tiny, then its square $(d\,x)^2$ is infinitely tiny compared even to $d\,x$, and it can be ignored: comparing $d\,x^2$ to $d\,x$, we have

$$\frac{d\,x^2}{d\,x} = \frac{2\,x\,d\,x + (\mathrm{dx})^2}{d\,x} = 2\,x + d\,x$$

Now 2x + d x is a finite number with an infinitesimal added on. This is just the same as 2 x (more on this later in this writeup).

This makes sense geometrically: x is a length, and the differences between squares (the **L** shape) is two lines. It is also seen in the double subtangent of the parabola (see RAD 59).

Similarly for cubes:



Here, the excess of adding on an infinitesimal layer of cube is

$$d\,x^3 \;=\; 3\,x^2\,d\,x \;+\; 3\,x\,(d\,x)^2 \;+\; 1\,(d\,x\,)^3$$

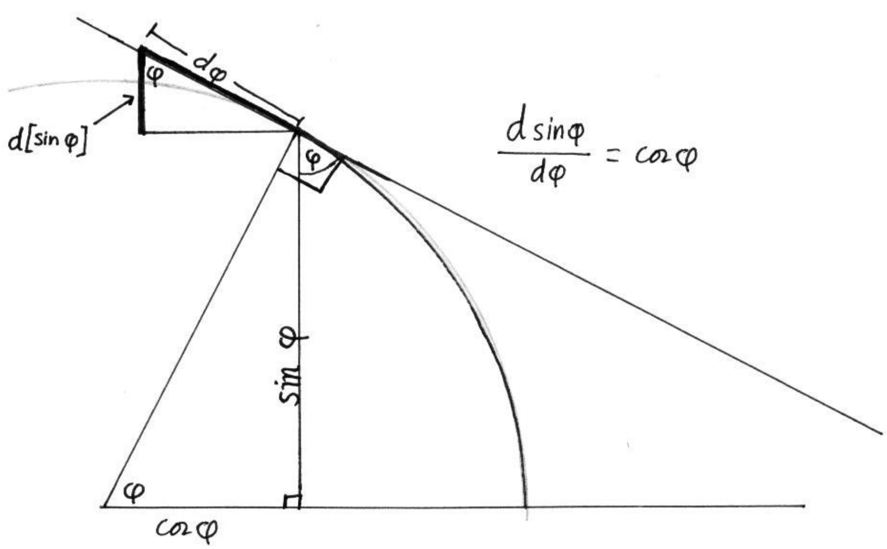
Which gives:

$$\frac{d\,x^3}{d\,x} \;=\; 3\,x^2 + \,3\,x\,(d\,x) \;+\; 1\,(d\,x)^2$$

Infinitesimals are so small, that they do not exist in regards to a finite number, so $3\,x^2 \;+\; 3\,x(d\,x)$ is just $3\,x^2$, so this becomes:

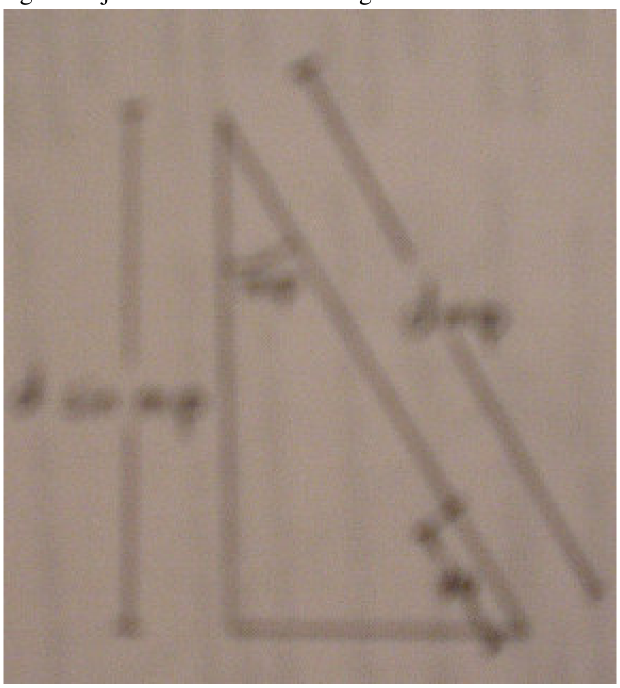
$$\frac{d\,x^3}{d\,x} \;=\; 3\,x^2.$$

Now, finally, let us look at what Gauss is using, the differential of a sine.



Here we have the sine for a particular angle ϕ, and a tangent to the circle at that spot. The small triangle coming up and to the left from the point on the circle is a differential triangle. Its hypothenuse is $d\,\phi$ because you actually measure angle by the circumference of the circle -- that's where "radians" come from, and the reason that $2\,\pi$ in radians is the same as 360° is that the circumference of going all the way around a circle is $2\pi r$. By continuing the tangent down to the horizontal axis, we make another triangle similar to our original one (this is similar to the semi-circle seen in the Archytas pedagogies), and this triangle is similar to our differential triangle. Therefore we can say that the upper angle of our differential triangle is also ϕ. So now, look at that differential triangle, and think of what $\frac{d \sin\phi}{d\phi}$ is. It's just cos ϕ! That is so simple, so why did we just learn it as a rule in class?!

Now, Gauss, in his differential has $\frac{d[\sin m\phi]}{d\phi}$ becoming $m\cos m\phi$. Where did that m coefficient come from? Here is a figure of just the differential triangle for $m = 4$:



(better picture coming up)

Here, you have $\frac{d \sin m\phi}{d\, m\phi} = \cos m\phi$, and since $\frac{d\, m\phi}{d\phi} = m$, we have $\frac{d \sin m\phi}{d\phi} = m\cos m\phi$ geometrically (since the proportion is changed m times), and also algebraically:

$$\frac{d\ \sin m\phi}{d\, m\phi} \cdot \frac{d\, m\phi}{d\phi} = \frac{d \sin m\phi}{d\phi} = \cos m\phi \cdot m = m\cos m\phi.$$

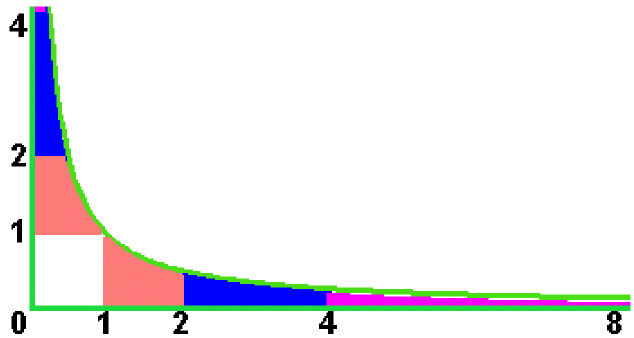
This is taught as the "chain rule," although no geometrical idea of why it is true is usually given.

So, now that we've put forward a (very brief) idea of what differentials are, let's get back to Gauss's proof!

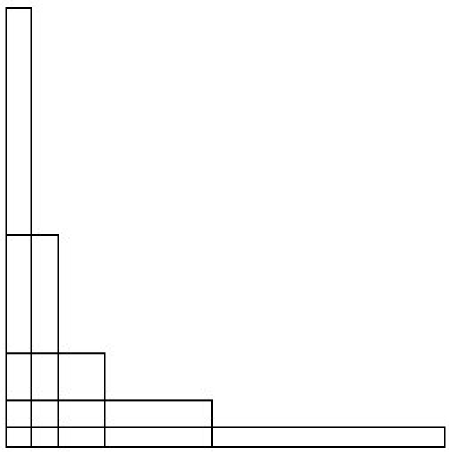
Another differential example. (This one will be useful in working on Gauss's paper on forces acting in inverse square of their distance -- the potential paper.)

Let's look at what $\frac{d[\frac{1}{x}]}{\mathrm{dx}}$ is. Let's start with this image from Bruce Director's Riemann for Anti-Dummies number 33 (www.wlym.com).

Figure 6



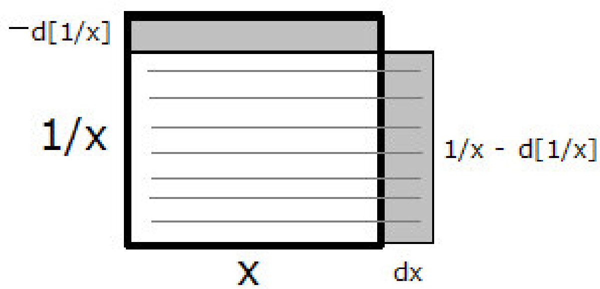
Bruce creates the shape of a hyperbola by making a series of rectangles of equal size.



In the above image, there are five rectangles:

1) a rectangle of height 4 and width 1/4
2) a rectangle of height 2 and width 1/2
3) a square of height and width = 1
4) a rectangle of height 1/2 and width 2
5) a rectangle of height 1/4 and width 4

So now let's look at the differential of 1/x:



Here, we have a rectangle (that looks a lot like a square - imagine x is nearly 1) in bold lines, with the new, differential rectangle extended to the right, and shrunk downwards (moving along the hyperbola). Now since all rectangles keeping the relationship of width=x and height=1/x have the same area (x · 1/x = 1), the gray area added to the right must be the same as the gray area removed from the top. Let's compare the areas of those two rectangles.

Top rectangle: width = x, height = d[1/x], area = x · d[1/x]
Right rectangle: width = dx, height = 1/x - d[1/x], area = dx · (1/x - d[1/x])

Now, setting those areas to be equal:

x · d[1/x] = dx · (1/x - d[1/x])

Now in the right-hand-side (RHS) of this equation, we have (1/x - d[1/x]). Now remember that differentials are infinitely small in comparison to finite quantities, because differentials are intentions, which are not comparable with sizes. Imagine a 14 year-old girl Samantha, who is 5'4" and still growing, and her 39 year old mother, also 5'4". Although the daughter may be 5'4" + d[Samantha] becuase she is going to be growing, she is still the same height as her mother. So (1/x - d[1/x]) is just (1/x) for us here. This leaves us with:

x · d[1/x] = dx · 1/x

which is

d[1/x] = dx · 1 / x^2

or

$$\frac{d[1/x]}{\mathrm{dx}} = \frac{1}{x^2}.$$

But remember that d[1/x] is negative when dx is positive, since the height of the rectangle shrinks while the width grows, so we add a negative sign and conclude:

$$\frac{d[1/x]}{\mathrm{dx}} = -\frac{1}{x^2}.$$

Another differential expression! Now go read the Gauss paper on potential at (http://www.wlym.com/~jross/curvature/). With this differential in hand, you should be able to get the differentials in section 2 (especially if you've worked on the 1827 Curved Surfaces Paper).