Section 18 of the 1799 FTA

Section 18

Gauss opens:

First of all, I observe that either curve is algebraic, and of order m if referred to orthogonal coordinates. If now the origin is assumed at C, the abscissa x taken in the direction of G, the corresponding y toward P, then x = r cos ϕ, y = r sin ϕ, and therefore generally for any n

$$r^n \sin n\,\phi = n\,x^{n-1}\,y - \frac{n\cdot(n-1)\cdot(n-2)}{1\cdot2\cdot3}\,x^{n-3}\,y^3 + \frac{n\cdot\ldots\cdot(n-4)}{1\cdot\ldots\cdot5}\,x^{n-5}\,y^5 - \text{etc.},$$
$$r^n \cos n\,\phi = x^n - \frac{n\cdot(n-1)}{1\cdot2}\,x^{n-2}\,y\,y + \frac{n\cdot(n-1)\cdot(n-2)\cdot(n-3)}{1\cdot2\cdot3\cdot4}\,x^{n-4}\,y^4 - \text{etc.}$$

Whoa! If you thought you were done with things from left field after sections 13 and 14, Gauss is hitting you again. It will take a bit of work on a number of different ideas before we can understand where these formulas came from, and it will demonstrate, again, the power of the complex domain in developing an understanding of magnitude.

Take a shower!

First off, take a shower! In fact, take three -- in one week. And make a schedule for taking three showers in a week. You could choose to combine Monday, Wednesday, and Friday, or perhaps Tuesday, Wednesday, and Sunday. Our understanding of cosines and sines depends on figuring out how many different choices of showering schedules we have.

One way to determine this, would be to write down all the combinations:

MTW	MWTh	MThF	MFS	MSSu
MTTh	MWF	MThS	MFSu	
MTF	MWS	MThSu		
MTS	MWSu			
MTSu				

(15 if you definitely have Monday)

TWTh	TThF	TFS	TSSu
TWF	TThS	TFSu	
TWS	TThSu		
TWSu			

(10 if you do not have Monday, but do have Tuesday)

WThF	WFS	WSSu
WThS	WFSu	
WThSu		

(6 if you have neither Mon nor Tues, but do have Wednesday)

ThFS	ThSSu
ThFSu	

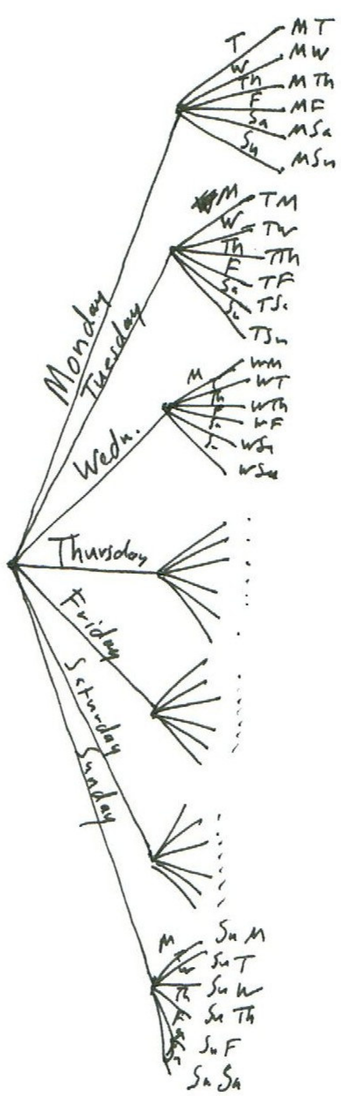
(3 if Thursday)

FSSu

(1 if you have none of Mon-Thurs)

This gives a total of 15 + 10 + 6 + 3 + 1 = 35, and it makes for a lot of work! If we were choosing 10 days out of a month, this would become *very* complex: do you think there is an easier way?

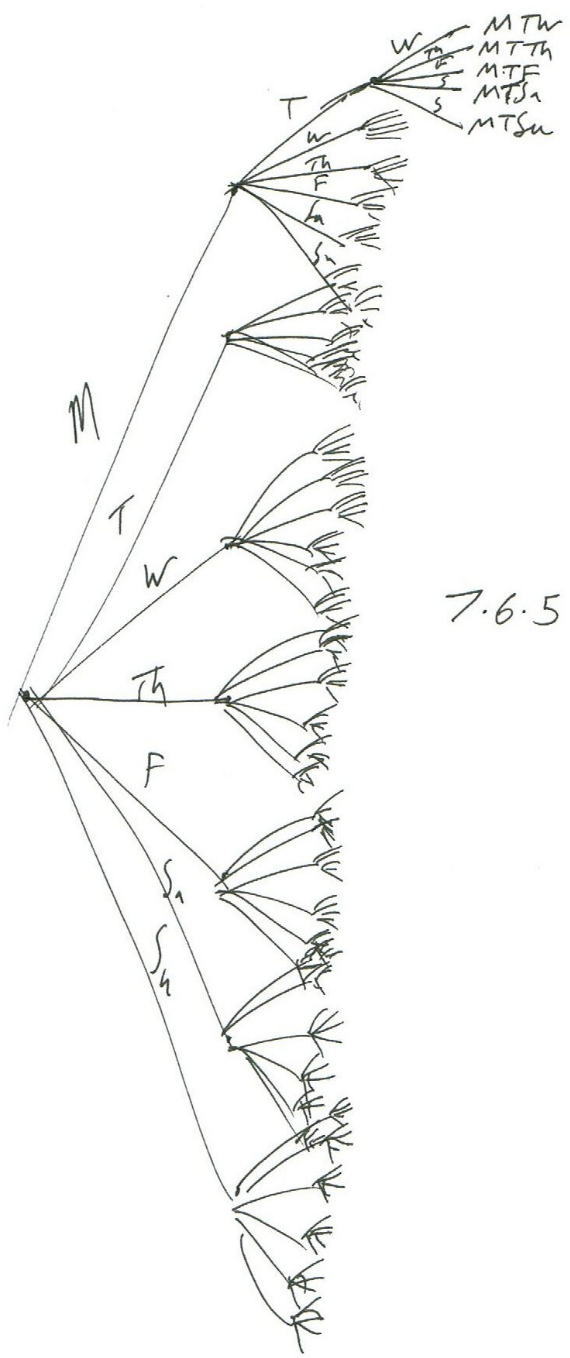
Let's try another approach: as you look at your empty weekly calendar, you have seven choices for the first day you will choose to take a shower on. M, T, W, Th, F, Sa, Su are all possibilities. For each initial day that you choose, you have six possibilities for the second day you choose to take a shower: if you chose Monday, you now can pick between T, W, Th, F, Sa, and Su; and if you chose Thursday, you can pick between M, T, W, F, Sa, and Su, and so forth. This can be pictured as a tree:



So we have 7•6 = 42 "leaves," or ways of having chosen one day, and then another.

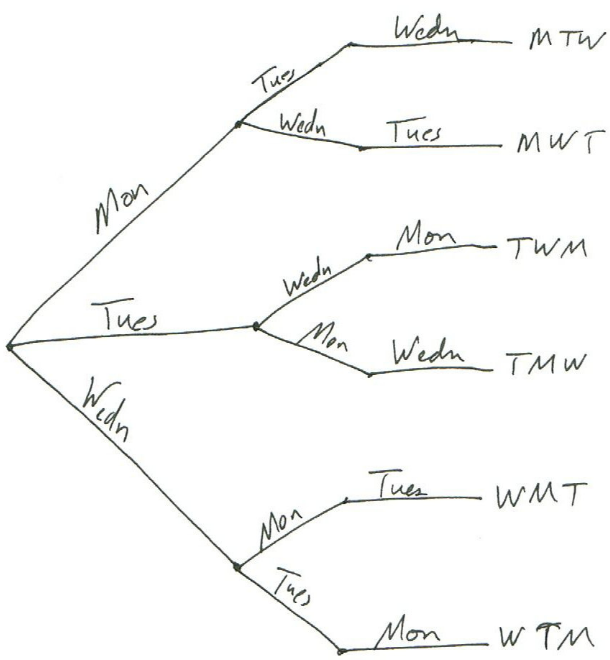
Take a break now, and think about how many ways we can choose two days to shower. Is it 42 ways? Think... The tree includes both M T and T M; Th Su and Su Th; W F and F W; and two of every possibility. Wouldn't we necessarily find each twice? So instead of 42, we must have 42 / 2 = 21 ways of picking two days to shower, once we consider duplicate schedules. Check it out if you like by figuring out all the combinations.

Now let's try looking at showering three days of the week. Add another layer of "stems" to our earlier tree. Each way of having chosen two days can have one of the five remaining possibilities for the third day: MT can choose W, Th, F, Sa, or Su; and WF can choose M, T, Th, Sa, or Su.



Now we have 7•6•5 = 210 leaves on our tree. But are there really that many ways of choosing three days of the week? How many leaves are duplicated? A Monday, Wednesday, Friday showerer would have leaves:
MWF, MFW, WMF, WFM, FMW, and FWM.

Six ways, but why six? Think of another, smaller tree. How many three-day leaves will have MWF in them? The leaf could start with a M, W, or F (three choices), followed by W or F (if M), M or F (if W), M or W (if F) -- two more choices. So we get three•two = six duplicate leaves.



Removing duplicates, this leaves us with 7·6·5 / 3·2 = 35 ways of choosing three shower days. We know from having written out the combinations that this is correct.

We have in fact devised a general way of choosing things. We make ourselves a fraction:

$$\frac{\text{number of leaves made on our choosing-tree}}{\text{number of ways of arranging the same thing (number of duplicate leaves)}}$$

So let's make the example of holding up two fingers:

$$\frac{5 \cdot 4}{2 \cdot 1}$$

There are 5 · 4 ways of holding up two fingers, if it matters in which order you held them up (thumb-ring and ring-thumb counting separately), which we must then divide by 2 · 1 because that is the number of ways of holding up the same two fingers in different orders.

Think of holding up fingers. How many ways can you hold up two fingers? How many ways can you hold *down* two fingers? How many ways can you hold up three fingers, and how many ways can you hold down three fingers? Do you know how many ways there are to arrange a four-day-a-week shower schedule?

Are these two fractions the same:

$\frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1}$ and $\frac{7 \cdot 6 \cdot 5 \cdot 4}{4 \cdot 3 \cdot 2 \cdot 1}$?

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

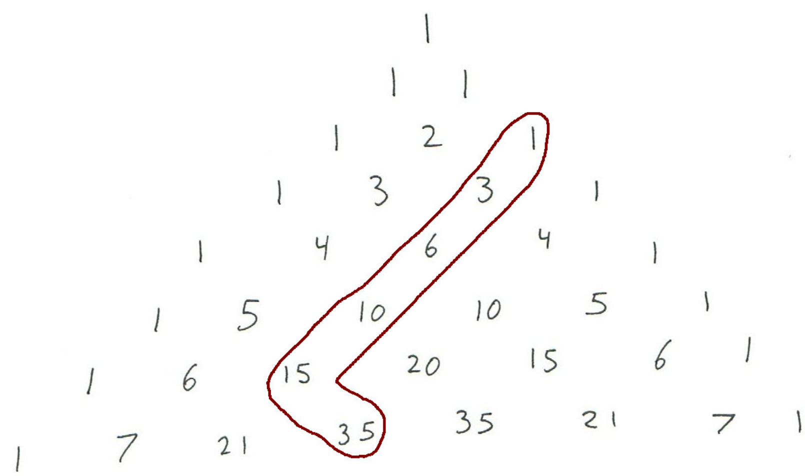
He is what Pascal called the "arithmetical triangle." If you haven't seen this before, try to figure out how it is constructed and what the next rows will be.

Now that you've figured out how the triangle is created, you'll be amazed that this triangle also is a shadow of the process of the art of combinations examined above. Look at the row
1 5 10 10 5 1
How many ways can you hold up no fingers? One finger? Two fingers? Three fingers? Four fingers? All five fingers?
And the row 1 7 21 35 35 21 7 1 has the 35 showering schedules for three days. But why? What does the repeated adding of numbers in one row to get the row below have to do with combinations?
Think: Let's come up with the number of ways you can choose 3 out of your five fingers. We'll show that this is exactly the same as adding the two numbers above it in the triangle, or, for our case, of looking at cases of holding up four fingers. Each combination of three fingers either will or will not include your thumb. Let's find out how many do include the thumb first -- once you've chosen your thumb, you have two fingers left to choose out of your remaining four -- and that can be done in six ways: (1 4 **6** 4 1). Now, if you did *not* use your thumb, you'd have to choose the three out of the remaining four fingers, which can be done in four ways (1 4 6 **4** 1). Adding the six and the four gives you the ten ways of choosing three of your five fingers (1 5 10 **10** 5 1). This reasoning wasn't specific to thumbs, and from it we can say that the entire triangle is created in a way that it corresponds to choosings.

From our investigation of trees from before, the general idea for any position of the triangle, say choosing n out of m (in the row begining with m), is:

$$\frac{(m)\cdot(m-1)\cdot(m-2)\cdot(m-3)\cdot\ldots\cdot(m-n)}{n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot 2\cdot 1}$$

There are plenty of patterns to find on the tree besides the one you figured out to get the next row (adding the two numbers above to get the number below). Do you remember writing out all the possible shower schedules for three per week? We got 15, 10, 6, 3, and 1 to add up to 35 (look back at the begining of this section). You have it here on Pascal's triangle!



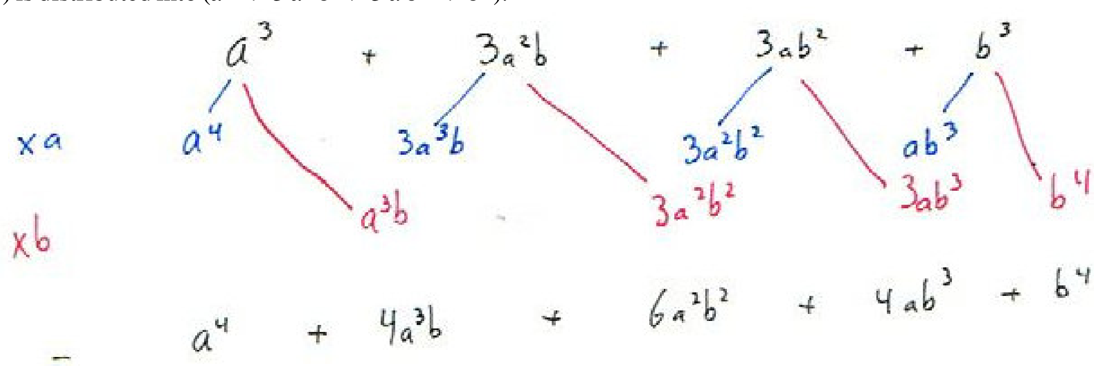
Now that we have joined the ideas of Pascal's triangle and the process of choosing, let's do something different: we'll look at multiplying factors.

Multiplying Factors

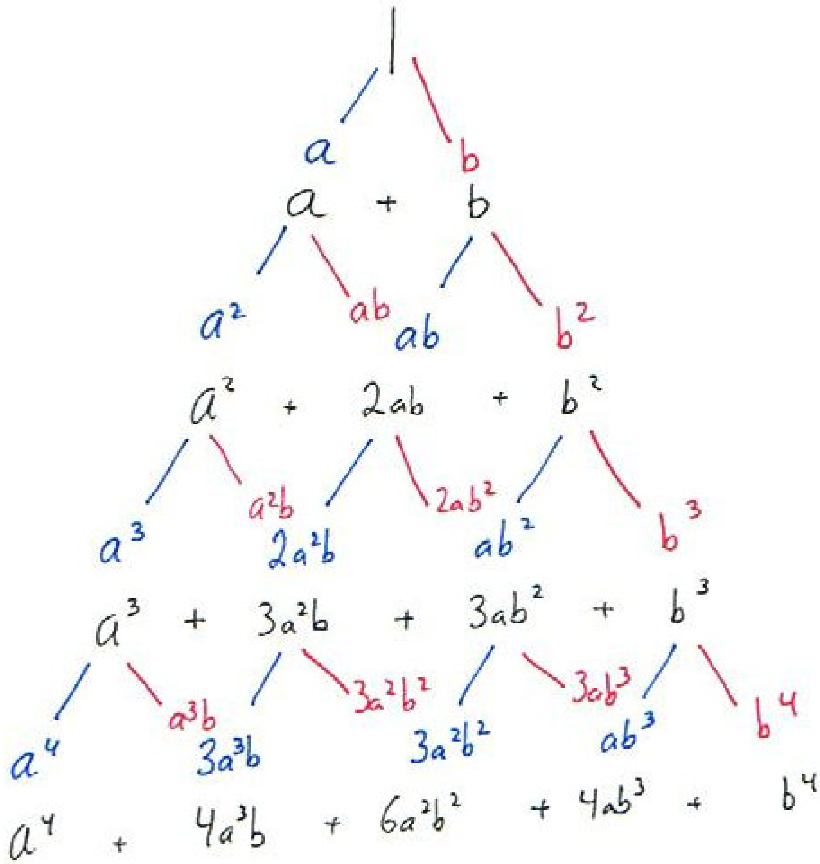
Take a look at what happens with multiplication (and work these through yourself!)

$(a + b)$	$a + b$
$(a + b)^2$	$a^2 + 2ab + b^2$
$(a + b)^3$	$a^3 + 3a^2b + 3ab^2 + b^3$
$(a + b)^4$	$a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
$(a + b)^5$	$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Aha, what do you notice about the coefficients? Now why is this? Let's take the case of the $(x+a)^3 \cdot (x+a) = (x+a)^4$. Let's look in particular at the way that the coefficients of $3ax^2$ and $3a^2x$ are added to get $6a^2x^2$. Think of the way that $(a + b)$ is distributed into $(a^3 + 3a^2b + 3ab^2 + b^3)$.



As you see, you get the same sort of relationship that we saw in choosing and Pascal's triangle, where the coefficient in the lower row is made by adding the terms to the upper-left and upper-right. Here is another diagram of this process of multiplication:



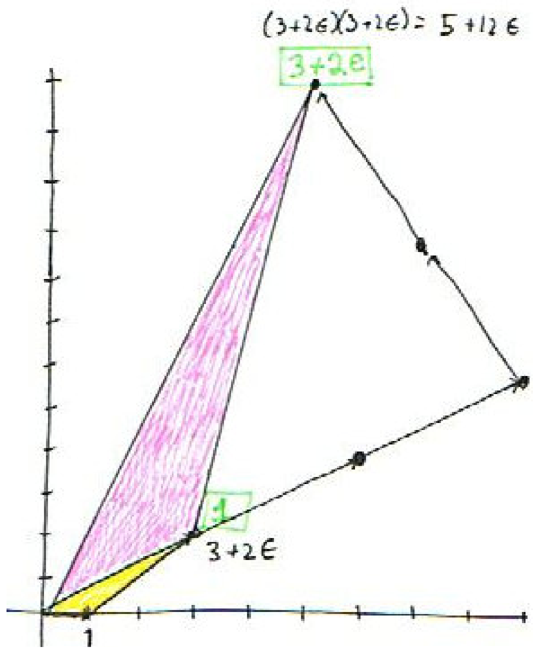
If you're wondering what choosing has to do with this multiplying, count up all the paths that led you downwards from 1 to $4\,a^3\,b$(I mean it -- count them up!) How many ways of *choosing* left or right led you to that $a^3\,b$term, and what does that have to do with its coefficient?

So now we have the triangle, choosing combinations, and multiplying factors all together as one idea -- once we combine this with the complex domain, the begining of section 18 will be clear.

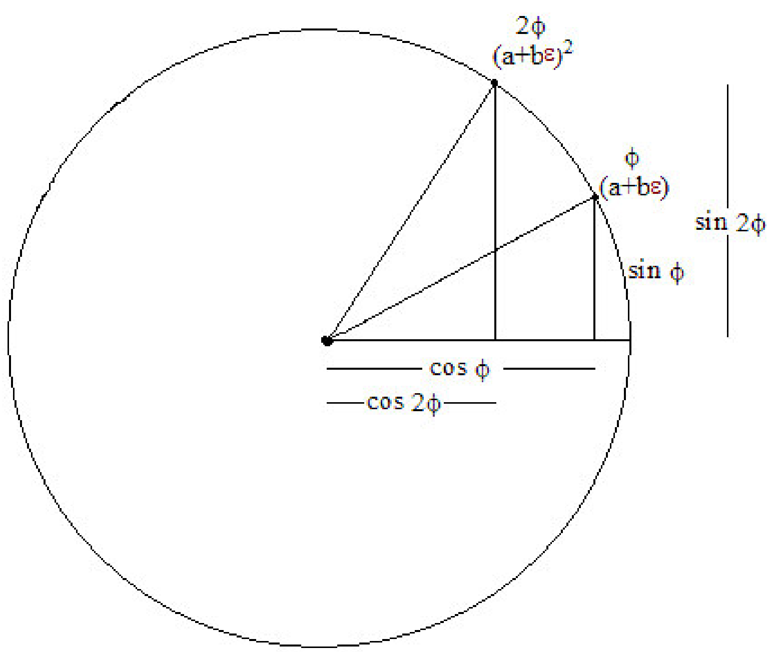
The Complex Domain

Let's bring all these ideas together. Using the standpoint of Wessel (see *A Source Book in Mathematics,* ed. Smith), we will revisit the multiplication of complex numbers to gain a fresh perspective on trigonometry. On a circle of radius 1, the multiplication of one number on this circle by another (or by itself) will leave us on the circle, because the lengths both being one, their product is also one. The angles of the two numbers will be added when they are multiplied. For example, look at the process of squaring a complex number on the unit circle.

On this circle, we have $a + b\sqrt{-1}$ or $a + b\,\epsilon$ as a particular location. When we multiply this number by itself, we find ourselves at twice its angle. So think about putting these two ideas together, the complex number multiplication and the angle doubling. First off, we better figure out exactly what $(a + b\,\epsilon)\cdot(a + b\,\epsilon)$actually is. We had developed FOIL earlier in looking at Gauss's paper, but the geometry that made sense there (areas), does not have the same intuitive meaning here with complex magnitudes. Let's look at this from the standpoint of what multiplication means -- think of "looking from someone else's standpoint" or the English possesive " 's." For example, if 2 were looking at what 1 would call 10, number 2 would call it 5. So 2's 5 is 10 from 1's standpoint. Looking similarly at $(a + b\epsilon)$ from $(a + b\epsilon)$'s standpoint, or $(a + b\epsilon)$'s $(a + b\epsilon)$, let's distribute. Here is an example of $(a + b\epsilon)$ looking at "its" $(a + b\epsilon)$: (This one good to see in color.)



Here we can literally "do" $(a + b\epsilon)$, a times, and then do $(a + b\epsilon)$, $b\epsilon$ times as well, as you see in the diagram with $(3 + 2\epsilon)$, getting us $(5 + 12\epsilon)$. The green $(3 + 2\epsilon)$ is $(3 + 2\epsilon)$ from the standpoint of the green, boxed 1. Work out why $(5 + 12\epsilon)$ is the product.



Using Pascal's triangle, we can easily find $(a + b\epsilon)^n$, getting the coefficients from the triangle, and using $(b\epsilon)$ instead of just (b). For example, $(a + b\epsilon)^2$ is $a^2 + 2ab\epsilon + (b\epsilon)^2$, which is $a^2 + 2ab\epsilon - b^2$. Now, think of $(a + b\epsilon)$ as $(\cos\phi + \epsilon\sin\phi)$. We can multiply $(\cos\phi + \epsilon\sin\phi)$ by itself to get

$$(\cos\phi)^2 + 2\,(\cos\phi)(\epsilon\sin\phi) + (\epsilon\sin\phi)^2,$$

which is

$$(\cos^2\phi\ +\ 2\,\epsilon\cos\phi\sin\phi\ -\ \sin^2\phi).$$

But, since this is at *double the angle,* it should also be equal $(\cos 2\phi + \epsilon\sin 2\phi)$. Since these two are the same, arrived at in two different ways (one algebraically and the other by thinking about the doubling of angle), we can say that:

$$(\cos 2\phi + \epsilon\sin 2\phi) = (\cos^2\phi\ +\ 2\,\epsilon\cos\phi\sin\phi\ -\ \sin^2\phi).$$

Separating these components into their direction (with or without ϵ), we get:

$(\cos 2\phi + \epsilon\sin 2\phi) = (\cos^2\phi\ -\ \sin^2\phi\ +\ 2\,\epsilon\cos\phi\sin\phi)$, which, by ϵ, splits into:

$\cos 2\phi = \cos^2\phi\ -\ \sin^2\phi$ and $\epsilon\sin 2\phi = 2\,\epsilon\cos\phi\sin\phi$.

Focusing on the ϵ part, we can divide by ϵ and have:

$$\sin 2\phi = 2\cos\phi\sin\phi.$$

Whoa, that is familiar! Do you remember deriving this in section 13 by construction? But now, it has come out of the multiplication all on its own. Read JBT's "From Cardan's Paradox to the Complex Domain" for the $\cos 2\phi$ part. Now quick, take this idea to any multiple of an angle, keeping in mind how Gauss opens section 18.

Back to Section 18

Gauss opens section 18 with two formulas:

$$r^n\sin n\phi\ =\ n^{x-1}\,y\ -\ \ldots$$

$$r^n\cos n\phi\ =\ x^n\ -\ \ldots$$

He is looking at cosines and sines of multiple angles. This is just what we are doing with multiplying. It's starting to make sense why we went through all of this. :-) Take the fourth row of Pascal's triangle, using $(b\,\epsilon)$:

$$a^4\ +\ 4\,a^3(b\,\epsilon)\ +\ 6\,a^2\,(b\,\epsilon)^2\ +\ 4\,a\,(b\,\epsilon)^3\ +\ (b\,\epsilon)^4$$

$$a^4\ +\ 4\,\epsilon\,a^3\,b\ -\ 6\,a^2\,b^2\ -\ 4\,\epsilon\,a\,b^3\ +\ b^4,$$

Separating out the ϵ parts from those without ϵ, to get the cosine and sine (ϵ) parts: we have:

$a^4\ -\ 6\,a^2\,b^2\ +\ b^4$ and $\epsilon(4\,a^3\,b\ -\ 4\,a\,b^3)$.

Returning to Gauss's paper, let's do this for what he has written, using $n\ =\ 4$, and $r = 1$ (since we are using a unit circle):

$$r^4\sin 4\,\phi\ =\ 4\,x^{4-1}\,y\ -\ \frac{4\cdot(4-1)\cdot(4-2)}{1\cdot 2\cdot 3}\,x^{4-3}\,y^3\ +\ \frac{4\cdot(4-1)\cdot(4-2)\cdot(4-3)\cdot(4-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}\,x^{4-5}\,y^5\ -\ \text{etc.},$$

$$r^4\cos 4\,\phi\ =\ x^4\ -\ \frac{4\cdot(4-1)}{1\cdot 2}\,x^{4-2}\,y\,y\ +\ \frac{4\cdot(4-1)\cdot(4-2)\cdot(4-3)}{1\cdot 2\cdot 3\cdot 4}\,x^{4-4}\,y^4\ -\ \text{etc.}$$

Which is:

$$r^4\sin 4\,\phi\ =\ 4\,x^3\,y\ -\ \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3}\,x\,y^3\ +\ \frac{4\cdot 3\cdot 2\cdot 1\cdot 0}{1\cdot 2\cdot 3\cdot 4\cdot 5}\,x^{-1}\,y\ -\ \text{etc.},$$

$$r^4\cos 4\,\phi\ =\ x^4\ -\ \frac{4\cdot 3}{1\cdot 2}\,x^2\,y^2\ +\ \frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4}\,x^0\,y^4\ -\ \text{etc.}$$

Note that the third term of the sine equation has "·0" in it. Since it is multiplied by zero, it is just zero. You get the same if you try to finish out the "etc." Dividing the fractions, we finally get:

$$r^4\sin 4\,\phi\ =\ 4\,x^3\,y\ -\ 4\,x\,y^3$$

$$r^4\cos 4\,\phi\ =\ x^4\ -\ 6\,x^2\,y^2\ +\ y^4$$

which is the same as what we got from Wessel:

$$a^4\ -\ 6\,a^2\,b^2\ +\ b^4$$

$$\epsilon(4\,a^3\,b\ -\ 4\,a\,b^3).$$

Now, since Gauss says that "$x\ =\ r\cos\phi,\ \ y\ =\ r\sin\phi$," we're doing exactly the same thing as above when we were looking at multiplying angles. So these formulas in section 18 of Gauss's paper, are easily understood from the complex domain, Wessel, and Pascal's triangle, but can you imagine figuring out what $(\cos 5\varphi)$ is by constructing it and measuring it out?