

## Understanding Section 13 of the 1799 F.T.A.

Lemma: Let  $m$  denote any positive whole number. Then the function  $\sin \phi x^m - \sin m\phi r^{m-1} x + \sin(m-1)\phi r^m$  is divisible by  $xx - 2\cos\phi rx + rr$ .

Proof: For  $m=1$ , the function is  $=0$  and therefore divisible by any factor. For  $m=2$ , the quotient is  $\sin \phi$ , and for any greater value the quotient will be  $\sin \phi x^{m-2} + \sin 2\phi rx^{m-3} + \sin 3\phi rrx^{m-4} + \text{etc.} + \sin(m-1)\phi r^{m-2}$ . It is easily confirmed that the product of this function multiplied by  $xx - 2 \cos \phi rx + rr$  is equal to the given function.

### *What's a lemma?*

Lemma means, "you'll see what I mean..." In sections 13 and 14, Gauss is making a shocking transition into his complex domain, but since he "considers it worth the trouble to show how" the fundamental theorem "can easily be elicited without their [imaginary numbers] help," he has to construct a shadow of the complex domain to allow his real proof to be expressible in non-complex terms. Imagine telling someone how to make a hammer before building telling them how to build the Grand Coulee Dam: that is what Gauss is doing here, so do not get frustrated if it is not clear where he is going. "Lemma tell ya, OK?"

### *What does he mean by "divisible by?"*

((Put in the Kästner on bases for numbers and long division.))

### *What does "+ etc. +" mean?*

Throughout the paper, Gauss writes "..." or "etc." in equations. This means that there is a pattern in what he is writing, and that the number of things could vary and they could not all be written out. 1, 2, 3, 4, ... 99, 100 for example. When in doubt, look at the first and last terms:

1, 2, 3, 4, ...  $n-1$ ,  $n$

is, when  $n$  is 3, just:

1, 2, 3

even though 4 was written originally (to make sure that you got the pattern).

*More on division (to be included in or be a substitute for the earlier section).*

In division, you have the following:

$$\text{Dividend} / \text{Divisor} = \text{Quotient.}$$

So in  $10 / 5 = 2$ , 10 is the dividend, 5 is the divisor, and 2 is the quotient.

In multiplication, you have

$$\text{Factor} \times \text{Factor} = \text{Product.}$$

To see whether a division is correct, you can multiply the quotient by the divisor to see if you indeed get the dividend, just as you could see that  $2 \times 5$  is indeed 10.

$$\text{Quotient} \times \text{Divisor} = \text{Dividend.}$$

## Gauss's proof in §13

*m=1*

Go ahead and put in 1 for m in our given function:

$$\sin \phi x^m - \sin m\phi r^{m-1} x + \sin (m-1)\phi r^m$$

putting in 1 for m gives us:

$$\sin \phi x^1 - \sin 1\phi r^{1-1} x + \sin (1-1)\phi r^1,$$

which is:

$$\sin \phi x - \sin \phi r^0 x + \sin 0\phi r.$$

$r^0$  is 1 (think about Bernoulli's spiral: what do you get if you do 0 rotation?) and the sine of  $0\phi$  is  $\sin 0$ , which is 0 (think of the circle), giving:

$$\sin \phi x - \sin \phi 1 x + 0$$

$$\sin \phi x - \sin \phi x$$

$$0.$$

Zero is divisible by anything, since anything goes into it zero times, so our function is divisible by our divisor.

$m=2$

First, let's put in 2 for m in our dividend:

$$\sin \phi x^2 - \sin 2\phi r^{2-1} x + \sin (2-1)\phi r^2,$$

which simplifies to:

$$\sin \phi x^2 - \sin 2\phi r^1 x + \sin 1 \phi r^2,$$

which is:

$$\sin \phi x^2 - \sin 2\phi rx + \sin \phi r^2.$$

Now, Gauss says that the quotient is  $\sin \phi$ , so let's multiply our divisor ( $xx - 2 \cos \phi rx + rr$ ) by our quotient to see if we get our given function:

$$\sin \phi (xx - 2 \cos \phi rx + rr).$$

When multiplying in this fashion, the factor "distributes" itself to all the terms in the other factor. Think of the example of reducing a photograph to half its size. Every object in the photograph will become half as large. The same thing happens here. Here we are applying a  $\sin \phi$  "reduction" to our  $xx - 2 \cos \phi rx + rr$  "photograph."

$$\sin \phi (xx - 2 \cos \phi rx + rr) = \sin \phi xx - \sin \phi x \cdot 2 \cos \phi rx + \sin \phi rr.$$

To see if Gauss is right about this, let's compare to see if the quotient x divisor really equals the dividend.

$$\sin \phi x^2 - \sin 2\phi rx + \sin \phi r^2 \stackrel{=?}{=} \sin \phi xx - \sin \phi x \cdot 2 \cos \phi rx + \sin \phi rr.$$

The first and last terms are no problem, both being  $\sin \phi x^2$  (or  $xx$ ) and  $\sin \phi r^2$  (or  $rr$ ), respectively. But look at the middle terms: they do not seem to be equal:

$$\sin 2\phi rx \stackrel{=?}{=} \sin \phi x \cdot 2 \cos \phi rx.$$

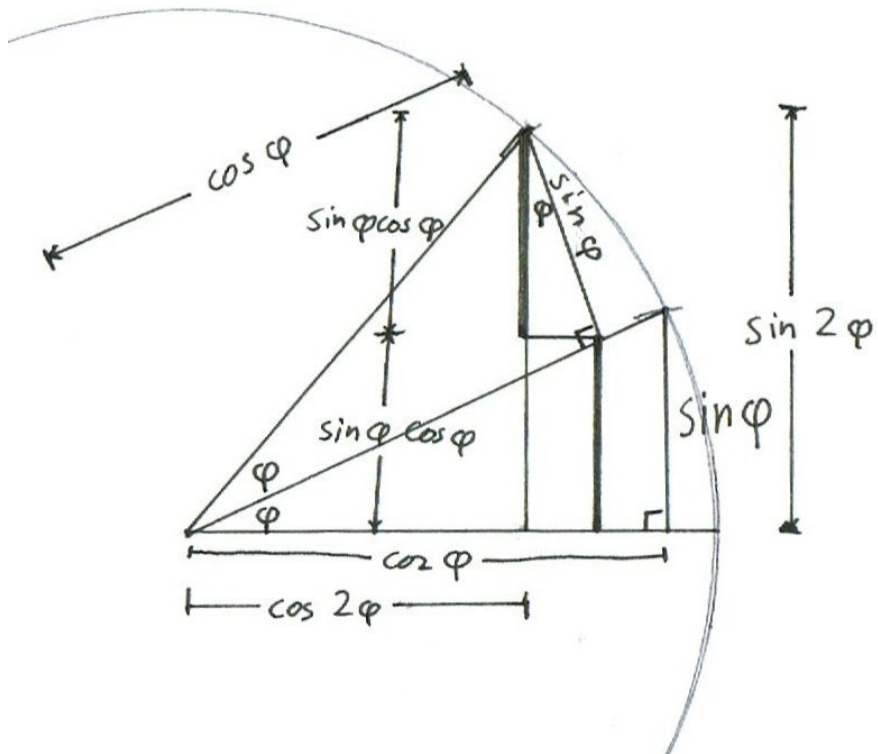
Now looking at the right hand side (RHS) of this, we can rearrange the multiplications to have  $2 \sin \phi \cos \phi rx$  instead of  $\sin \phi x \cdot 2 \cos \phi rx$ , so:

$$\sin 2\phi rx \stackrel{=?}{=} 2 \sin \phi \cos \phi rx,$$

and not worrying about the  $rx$  part, we have to figure out:

$$\sin 2\phi \stackrel{=?}{=} 2 \sin \phi \cos \phi,$$

which you can actually verify geometrically:



Here you have the  $\sin 2\phi$  being broken up into the two bolded vertical segments, each of which is  $\sin \phi \cos \phi$ . For an understanding of  $\cos \phi \sin \phi$  and related things, see Jonathan Tennenbaum's "From Cardan's Paradox to the Complex Domain."

### Greater values of $m$

Let's first look at what we get doing the multiplications for some greater values of  $m$ :

### ■ m=3

Dividend:

$$\sin \phi x^3 - \sin 3\phi r^{3-1} x + \sin (3-1)\phi r^3, \text{ which is:}$$

$$\sin \phi x^3 - \sin 3\phi r^2 x + \sin 2\phi r^3.$$

Quotient x Divisor:

We'll have to determine what the quotient is. Look at the first two and last two terms as it is written in Gauss's paper:

$$\sin \phi x^{3-2} + \sin 2\phi r x^{3-3} + \text{etc.} + \sin 3\phi r r x^{3-4} + \sin (3-1)\phi r^{3-2}$$

$$\sin \phi x + \sin 2\phi r + \text{etc.} + \sin 3\phi r r x^{-1} + \sin 2\phi r.$$

As you can see, the second-to-last term seems very strange, and the last term is the same as the second term: so the quotient has only two terms:  $(\sin \phi x + \sin 2\phi r)$ . Multiplying the quotient by the divisor, we have:

$$(\sin \phi x + \sin 2\phi r) x (xx - 2 \cos \phi r x + rr).$$

If the multiplication is giving you trouble, think of having three-and-a-half of 10. That means  $3 \times 10$  and  $\frac{1}{2} \times 10$  added together. So we will do:

$$(\sin \phi x + \sin 2\phi r) x (xx - 2 \cos \phi r x + rr).$$

**$\sin \phi x$**  x  $(xx - 2 \cos \phi r x + rr)$  +  **$\sin 2\phi r$**  x  $(xx - 2 \cos \phi r x + rr)$ , expanding to:

$$\sin \phi xxx - 2 \sin \phi \cos \phi rxx + \sin \phi rrx + \sin 2\phi rxx - 2 \sin 2\phi \cos \phi rrx + \sin 2\phi rrr.$$

Now, since  $2 \sin \phi \cos \phi = \sin 2\phi$ :

$$\sin \phi xxx - \underline{\sin 2\phi rxx} + \sin \phi rrx + \underline{\sin 2\phi rxx} - 2 \sin 2\phi \cos \phi rrx + \sin 2\phi rrr.$$

The underlined terms cancel each other, leaving:

$$\sin \phi x^3 + \sin \phi r^2 x - 2 \sin 2\phi \cos \phi r^2 x + \sin 2\phi r^3.$$

Comparing dividend and quotient x divisor:

$$\sin \phi x^3 - \sin 3\phi r^2 x + \sin 2\phi r^3 \stackrel{=?}{=} \sin \phi x^3 + \sin \phi r^2 x - 2 \sin 2\phi \cos \phi r^2 x + \sin 2\phi r^3.$$

The first and last terms are the same, leaving us with just:

$$- \sin 3\phi r^2 x \stackrel{=?}{=} \sin \phi r^2 x - 2 \sin 2\phi \cos \phi r^2 x.$$

Note that all the terms include  $r^2 x$ , which we don't have to concern ourselves with. Five's being three plus two does not depend on what it is that we have five, three, and two of. Ignoring  $r^2 x$ , (or dividing everything by it,) leaves us with:

$$- \sin 3\phi \stackrel{=?}{=} \sin \phi - 2 \sin 2\phi \cos \phi,$$

which, adding  $\sin 3\phi$  to both sides, becomes:

$$0 \stackrel{=?}{=} \sin 3\phi - 2 \sin 2\phi \cos \phi + \sin \phi.$$

This looks more difficult to figure out!

## *An unanswered question*

Let's not try do determine if this is true just yet. Instead, keep trying higher values of m to see what sorts of patterns we come up with. Remember, Gauss is saying that it is "easily confirmed" that the quotient times the divisor give us our given function (dividend). Press on!

■ **m=4**

We get for the dividend:

$$\sin \phi x^4 - \sin 4\phi r^3 x + \sin 3\phi r^4,$$

and for the quotient:

$$\sin \phi x^2 + \sin 2\phi rx + \sin 3\phi r^2.$$

(Figure out on your own why there are three terms in the quotient.) Our quotient times our divisor is:

$$(\sin \phi x^2 + \sin 2\phi rx + \sin 3\phi r^2) \times (xx - 2 \cos \phi rx + rr), \text{ which is}$$

$$\mathbf{\sin \phi x^2} \times (xx - 2 \cos \phi rx + rr) + \mathbf{\sin 2\phi rx} \times (xx - 2 \cos \phi rx + rr) + \mathbf{\sin 3\phi r^2} \times (xx - 2 \cos \phi rx + rr).$$

(Think of a three-by-three FOIL-type grid if this multiplication doesn't make sense from what's been said.) This gives us:

$$\begin{aligned} & \sin \phi x^4 - 2 \sin \phi \cos \phi rx^3 + \sin \phi r^2 x^2 \\ & + \sin 2\phi rx^3 - 2 \sin 2\phi \cos \phi r^2 x^2 + \sin 2\phi r^3 x \\ & + \sin 3\phi r^2 x^2 - 2 \sin 3\phi \cos \phi r^3 x + \sin 3\phi r^4. \end{aligned}$$

Now look at our product in a grid:

$$\begin{array}{r} \sin \phi x^4 \qquad - 2 \sin \phi \cos \phi rx^3 \qquad + \sin \phi r^2 x^2 \\ + \sin 2\phi rx^3 \qquad - 2 \sin 2\phi \cos \phi r^2 x^2 \qquad + \sin 2\phi r^3 x \\ + \sin 3\phi r^2 x^2 \qquad - 2 \sin 3\phi \cos \phi r^3 x \qquad + \sin 3\phi r^4 \end{array}$$

And you'll note that terms that have the same powers of r and x variables appear as "/"-shaped diagonals on the grid. Comparing the dividend with the product of the quotient x divisor, we see that the first and last terms of the dividend and the product ( $\sin \phi x^4$  and  $\sin 3\phi r^4$ ) are the same and cancel.

Look at the first diagonal next to  $\sin \phi x^4$  :

$$\sin 2\phi rx^3 - 2 \sin \phi \cos \phi rx^3.$$

Based on our trigonometry work from m=2, we know that these terms subtract each other to give us zero. The next diagonal is (leaving out the  $r^2 x^2$  common to all):

$$\sin 3\phi - 2 \sin 2\phi \cos \phi + \sin \phi.$$

This is the return of our unanswered question from m=3! For the m=3 case to be correct, the diagonal must also be zero.

That is:

$$\sin 3\phi - 2 \sin 2\phi \cos \phi + \sin \phi \text{ must be } =0.$$

This leaves as the only thing that doesn't disappear in the product:

$$\sin 2\phi r^3 x - 2 \sin 3\phi \cos \phi r^3 x,$$

and the only thing that doesn't disappear in the dividend:

$$- \sin 4\phi r^3 x.$$

So for m=4, it is necessary that

$$\sin 2\phi r^3 x - 2 \sin 3\phi \cos \phi r^3 x = - \sin 4\phi r^3 x,$$

or, forgetting about  $r^3 x$  and adding  $\sin 4\phi$  to both sides:

$$\sin 4\phi - 2 \sin 3\phi \cos \phi + \sin 2\phi = 0$$

must be true, if Gauss is correct in this proof.

## The general case

Compare the unresolved questions for  $m=3$ ,  $m=4$ , and  $m=5$  (which we didn't go over here, but you can do on your own, along with trying out  $m=6$ ). Each higher power of  $m$  depends on all the earlier unresolved questions and introduces a new one as well.

$$\sin 3\phi - 2 \sin 2\phi \cos \phi + \sin \phi =? = 0$$

$$\sin 4\phi - 2 \sin 3\phi \cos \phi + \sin 2\phi =? = 0$$

$$\sin 5\phi - 2 \sin 4\phi \cos \phi + \sin 3\phi =? = 0$$

$$\sin 6\phi - 2 \sin 5\phi \cos \phi + \sin 4\phi =? = 0$$

A clear pattern! Can you figure out a way of writing the general case? Do it before reading on!

$$\sin A\phi - 2 \sin (A-1)\phi \cos \phi + \sin (A-2)\phi =? = 0.$$

Reorganize this to get:

$$\sin A\phi = 2 \sin (A-1)\phi \cos \phi - \sin (A-2)\phi.$$

Splitting up  $2 \sin (A-1)\phi \cos \phi$  into  $\sin (A-1)\phi \cos \phi + \sin (A-1)\phi \cos \phi$ , and subtracting from both sides, we get:

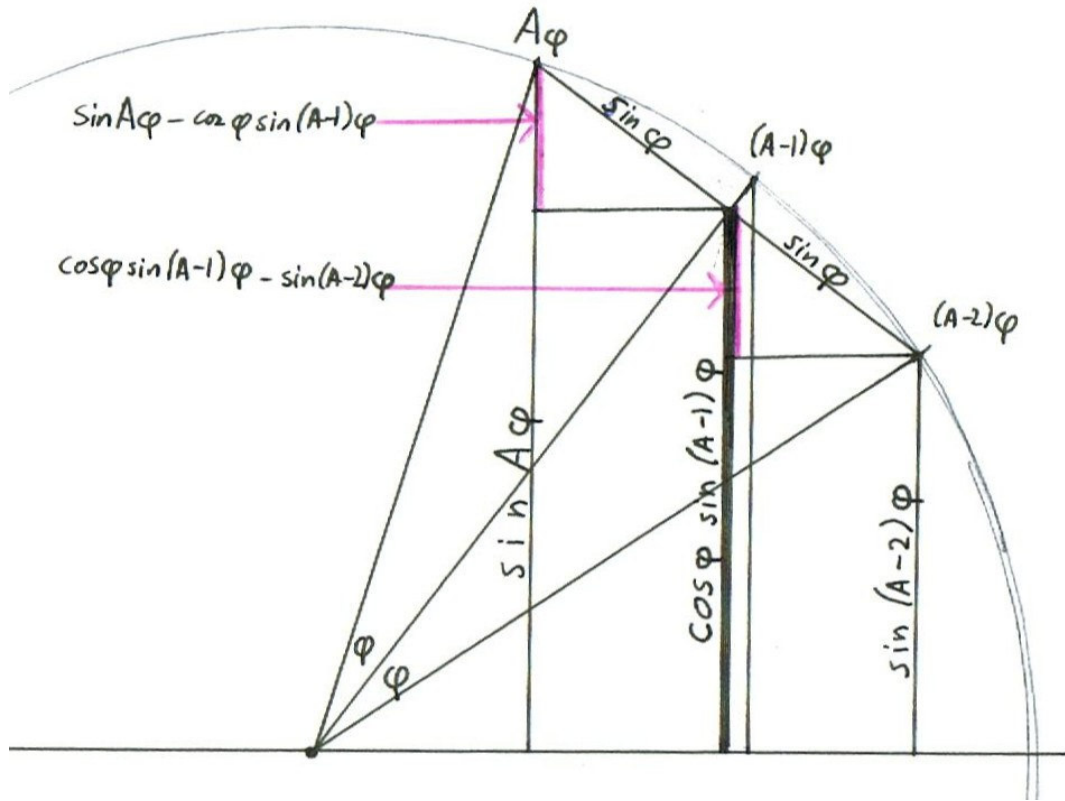
$$\sin A\phi = \sin (A-1)\phi \cos \phi + \sin (A-1)\phi \cos \phi - \sin (A-2)\phi$$

$$\sin A\phi - \sin (A-1)\phi \cos \phi = \sin (A-1)\phi \cos \phi - \sin (A-2)\phi$$

or

$$\sin A\phi - \cos \phi \sin (A-1)\phi = \cos \phi \sin (A-1)\phi - \sin (A-2)\phi$$

This we will prove geometrically:



Mark off angles  $A\phi$ ,  $(A-1)\phi$ , and  $(A-2)\phi$  on the circle. We can get  $\sin A\phi$  and  $\sin(A-2)\phi$  easily. To get the  $\cos \phi \sin(A-1)\phi$

piece, connect the points on the circumference at  $A\phi$  and  $(A-2)\phi$ , cutting the line from the origin to the  $(A-1)\phi$  spot on the circumference. The triangle formed from the origin, the  $A\phi$ , and the cut spot on  $(A-1)\phi$  is a right triangle (figure out why!), and has angle  $\phi$ . That makes the base of the triangle have length  $\cos \phi$ . Now revisit the  $(A-1)\phi$  angle. The line dropped directly down is  $\sin(A-1)\phi$ . Now that we have cut the radius of the circle going out to  $(A-1)\phi$  to have a length of just  $\cos \phi$ , the line dropped from this cut will have a length of  $\cos \phi \sin(A-1)\phi$ . Think: instead of "1's  $\sin(A-1)\phi$ " it is rather "( $\cos \phi$ )'s  $\sin(A-1)\phi$ ." Multiplying is like the possessive. (See Tennenbaum's "Cardan's Paradox to the Complex Domain" or the Complex Multiplication Pedagogy (not yet written) to help you with this if it seems unclear.)

Now we have  $\sin A\phi$ ,  $\sin(A-2)\phi$ , and  $\cos \phi \sin(A-1)\phi$ . Drawing horizontals from  $(A-2)\phi$  to  $\cos \phi \sin(A-1)\phi$  and from  $\cos \phi \sin(A-1)\phi$  to  $A\phi$ , we create the pink segments which are  $[\sin A\phi - \cos \phi \sin(A-1)\phi]$  and  $[\cos \phi \sin(A-1)\phi - \sin(A-2)\phi]$ . Ah! Now we have a geometric understanding of what the general case for section 13 is. We needed to know that  $\sin A\phi - \cos \phi \sin(A-1)\phi = \cos \phi \sin(A-1)\phi - \sin(A-2)\phi$ , so it comes to determining whether the pink segments are of equal size.

Reflection (literally) gives us to realize that the hypotenuses of the two triangles formed with pink sides are both  $\sin \phi$ . Since the triangles are at the same angle (being formed by the same line as hypotenuse), the pink (left) sides must be of equal length, meaning that we've demonstrated section 13!

To review, the equality of the pink sides let us know that  $\sin A\phi - \cos \phi \sin(A-1)\phi = \cos \phi \sin(A-1)\phi - \sin(A-2)\phi$ , which was the general form for the required trigonometric identity required by the higher values of  $m$ . We proved  $m=1$  and  $m=2$ , so we can now consider Gauss's section "easily demonstrated," although perhaps not so easily.