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CLEARY TRANS. RIEMANN

On the Representability of a Function by means of
A TRIGNOMETRIC FUNCTION

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ON THE REPRESENTABILITY OF A FUNCTION BY MEANS OF A TRIGONOMETRIC SERIES (From the 13th volume of the Proceedings of the Royal Society of Sciences at Goettingen)#

This treatise was submitted by the author in 1854 for his Habilitation in the Faculty of Philisophy at the University of Goettingen. Although the author did not, as it seems, intend its publication, its publication in a totally unchanged form appears to be sufficiently justified, as is the great interest in the subject matter itself, because of its method of handling the most important principles of infinitesimal analysis.

Braunschweig July 1867 R. Dedekind

The following essay on trigonometric series consists of two essentially different sections. The first section contains a history of the research and opinions on arbitrary (graphically given) functions and on their representability by means of a trigonometric series. In ~~[illegible]~~ compiling it I was allowed to use some suggestions from the famous mathematician, to whom we are indebted for the first basic wrok in this subject ~~[illegible]~~. In the second part

CONDUCTING
(~~[illegible]~~ AN INVESTIGATION)

I will be ~~[illegible]~~ into the representability of a function by means of a trigonometric series which will also include cases that have been uncompleted up to now. It was necessary to preface this with a short essay on the concept of a DETERMINATE integral and on the extent of its validity.

THE HISTORY OF THE QUESTION OF THE REPRESENTABILITY OF AN ARBITRARILY GIVEN FUNCTION BY MEANS OF A TRIGONOMETRIC SERIES.

I.

The Trigonometric series called that by Fourier, i. e., the series of the form

$$\begin{aligned} & a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots \\ + \tfrac{1}{2} b_0 + {} & b_1 \cos x + b_2 \cos 2x + b_3 \cos 3x + \cdots \end{aligned}$$

play an important role in that very section of mathematics where totally arbitrary functions occur. Yes, we can even assert (ON PRINCIPLE) that the most essential progress in this section of mathematics that is so important for physics is dependent on a clearer insight into the nature of these series. Even in the first mathematical research which led to the observation of arbitrary functions, the question was already discussed, whether such a totally arbitrary function could be expressed through a series of the forms mentioned above.

This happened in the middle of the previous century, during the opportunity occasioned by INVESTIGATIONS into vibrating chords, which engaged the most famous mathematicians of that time. Naturally, we cannot present their views about our subject, without going into this problem.

ARE APPROXIMATELY THOSE IN

REALITY, it is well known that the form of a vibrating chord that is stretched in one plane becomes determined by means of the partial differential equation $\frac{\partial^2 y}{\partial t^2} = \alpha\alpha \frac{\partial^2 y}{\partial x^2}$ if x is the distance of one of the chord's undetermined points from its starting position, y is its distance from its position of rest at time t, [AND WHERE] a is independent of t, and of x too, given a chord that is equally thick everywhere.

The first person who gave a GENERAL solution to this equation was d'Alembert.

He showed# that every function of x and t, which is established for y in order to make the equation [INTO AN] identical [ONE] must be contained in the form $f(x+\alpha t) + \varphi(x-\alpha t)$ which follows from the introduction of the independent variable magnitude $x+\alpha t, x-\alpha t$ instead of x, t. As a result $\frac{\partial^2 y}{\partial x^2} - \frac{1}{\alpha\alpha}\frac{\partial^2 y}{\partial t^2}$ turns into $4\frac{\partial \frac{\partial y}{\partial(x+\alpha t)}}{\partial(x-\alpha t)}$.

Except for this [PARTIAL] differential equation, which follows from the GENERAL laws of motion, y still has to fulfill the condition in which the chord's fixed points are always = 0. So, when x = 0 in one of these points, and $x = l$ in the other,

$f(\alpha t) = -\varphi(-\alpha t)$, $f(l+\alpha t) = -\varphi(l-\alpha t)$ and consequently

$$f(s) = -\varphi(-s) = -\varphi(l-(l+s)) = f(2l+s),$$
$$y = f(\alpha t + x) - f(\alpha t - x).$$

#Memoires de l'academie de Berlin. 1747. pag. 214

After d'Alembert had offered this as the ~~[illegible]~~ GENERAL solution for the problem, he occupied himself with the equation $f(z) = f(2l + z)$ in a continuation# of his treatise, i.e., he ~~[illegible]~~ searched for

Ibid. pag. 220.

analytic expressions which remain unchanged when z increases to $2l$.

It was to Euler's essential credit, who gave a new presentation of this work [DONE] by d'Alembert in the following year's edition of the Berlin Proceedings## that he recognized more accurately the

Memoires de l'Academie de Berlin 1748. pag. 69.

essence of the conditions which function f(z) must satisfy. He noticed that the nature of the problem would be totally determined by the motion of the chord if at any point in time the form of the chord and the velocity for every point (thus y und $\frac{\partial y}{\partial t}$) are given. He also showed that when we consider both of these functions to be determined by arbitrarily drawn curves, the d'Alembertian function f(x) can always be found by means of a simple geometrical construction. Actually, assuming that $t = 0$, $y = g(x)$ und $\frac{\partial y}{\partial t} = h(x)$ then we get $f(x) - f(-x) = g(x)$, $f(x) + f(-x) = \frac{1}{a}\int h(x)\,dx$ for the values of x between 0 and l, and consequently, we get function f(z) between $-l$ und l. What follows from this is the function's value for every other value of x by means of the equation. [$f(z) = f(2l + z)$.] This is Euler's definition of function f(x) presented in abstract, but now more generally current concepts.

D'Alembert immediately### protested against this extension of

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his method by Euler, because his method necessarily presupposed that y can be analytically expressed in t and x.

Before an answer ensued from Euler about this, a third treatment of this subject appeared from Daniel Bernoulli# that

Memoires de l'academie de Berlin 1753. p. 147

was totally different from these two. Taylor had seen even before d'Alembert that $\frac{\partial^2 y}{\partial t^2} = aa\frac{\partial^2 y}{\partial x^2}$ at the same time that y is 0 (FOR $x = 0$ AND) always equal to 0 for $x = l$ when we have $y = \sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$ AND PLACE (HERE FOR N.) a whole number. It was from this that he explained the physical fact that a chord can also give the key notes for a chord 1/2, 1/3, 1/4... as long (the remaining notes are created in the same way) besides giving its own key note. Therefore, he held his particular solution (AS THE GENERAL ONE) i.e., he believed that the chord's vibration would always be at least very closely expressed by the equation if the whole number n were determined according to the pitch of the tone. The observation that a chord can produce its various tones simultaneously now led Bernoulli to the comment that the chord (according to theory) can also vibrate accoring to the equation $y = \Sigma a_n \sin\frac{n\pi x}{l} \cos\frac{n\pi a}{l}(t - \beta_n)$.

He maintained that this equation was the most [GENERAL]### because because all [OBSERVED MODIFICATIONS] of the phenomenas could be explained from it. In order to support this VIEW, he investigated the vibrations of a stretched thread that had no mass, and which was WEIGHTED down with finite masses placed on it AT particular points. He showed that the masses' vibrations can always be reduced into a quantity for those vibrations that is equal to the number of points, given that each of those vibrations lasts an equal length of time for all masses.

l.c. p. 157. art. XIII

This work by Bernoulli induced a new article by Euler which was printed immediately following Bernoulli's in the Proceedings of the Berlin Academy. In it he firmly maintains against d'Alembert that function f(z) can be entirely arbitrary between the limits of +l and −l. He also noted that Bernoulli's solution (which he had already presented as a special one) is therefore GENERAL and, naturally, is only GENERAL when the series $a_1 \sin\frac{x\pi}{l} + a_2 \sin\frac{2x\pi}{l} + \dots + \frac{1}{2}b_0 + b_1 \cos\frac{x\pi}{l} + b_2 \cos\frac{2x\pi}{l} + \dots$ can represent the ordinates of a totally arbitrary curve between the abscissas 0 and for abscissa x. Now at that time no one doubted that all the conversions that could be undertaken with an analytical expression, whether it was finite or infinite, were valid for every value of THE undetermined magnitudeS or were only inapplicable in very special cases. Therefore it appeared impossible to present an algebraic curve, or generally, an analytically given non-periodic curve, by means of the expressions presented above. So Euler believed that the question had to be decided against Bernoulli.

Meanwhile the dispute between Euler and d'Alembert was still unresolved. The former induced a young, and at that time still little known mathematician, Lagrange, to attempt to find a solution to the problem in an entirely new way through which he would succeed in getting Euler's results. He undertook to determine the vibrations of a thread that had no mass and which was WEIGHTED down with a finite, INDETERMINATE number of equally large masses in equally spaced intervals, and then he investigated how this vibration changed as the number of masses increased UNTO THE INFINITE. Despite the adroitness and the lavish use of analytical devices with which he carried out the first section of this investigation, the transition from finite to infinite nevertheless left so much to be desired that d'Alembert could continue to vindicate his solution as having the glory of the greatest GENERALITY, which he did in a peice placed at the beginning of his opuscles mathematica. Therefore the opinions of the most famous mathematicians of that time were divided and remained divided about this subject: for even in later works everyone essentially stood by their position.

So in order to finally compare the VIEWS about arbitrary functions and about their representability by means of a trigonometric series that had developed from the opportunity this problem presented, Euler first of all INTRODUCED these functions into Analysis and, basing himself on a geometric mode of perception, used infinitesimal calculations on them. Lagrange held Euler's results (his geometrical constructions of the vibration's course) to be correct: but he was not pleased with Euler's geometric treatment of these functions. D'Alembert, in comparison, agreed with Euler's method of comprehending differential equations and limited himself to attacking the accuracy of Euler's results, because one could