In a preceding part of this article, we have obtained the equation

$$0 = 4\pi\rho + \frac{\partial \bar{U}}{\partial w},$$

which combined with $0 = \nabla^2(\partial\bar{U}/\partial w)$ gives

$$0 = \nabla^2\rho,$$

and therefore the density ρ induced on any element $d\sigma$, which is evidently a function of the coordinates x, y, z of p, is also [harmonic]; it is moreover evident that ρ can never become infinite when p is within the surface.

It now remains to prove that the formula

$$V = \frac{1}{4\pi}\int d\sigma\, \bar{V}\,\frac{\partial \bar{U}}{\partial w} = -\int d\sigma\,\rho\bar{V},$$

[gives a function V within the surface which always tends to the limit $\bar{V}$ as the surface is approached].[4]

For this, suppose the point p to approach infinitely near the surface; then it is clear that the value of ρ, the density of the electricity induced by p, will be insensible, except for those parts infinitely near to p, and in these parts it is easy to see that the value of ρ will be independent of the form of the surface, and depend only on the distance p, $d\sigma$. But we shall afterwards show (art. 10), that when this surface is a sphere of any radius whatever, the value of ρ is

$$\rho = \frac{-\alpha}{2\pi f^3},$$

α being the shortest distance between p and the surface, and f representing the distance p, $d\sigma$. This expression will give an idea of the rapidity with which ρ decreases, in passing from the infinitely small portion of the surface in the immediate vicinity of p, to any other part situate at a finite distance from it . . . . It is also evident that the function V, determined by the above-written formula, will have no singular values within the surface under consideration.

[The author continues by proving the formula expressing the symmetry of Green's function.]

65. *Gauss on Potential Theory*[1]

19.

Let V be the potential of a system of masses M_1, M_2, M_3, . . . , located at the points P_1, P_2, P_3, . . .; v the potential of a second system of masses m_1, m_2, m_3, . . . , located at the points p_1, p_2, p_3, . . .; furthermore, let V_1, V_2, V_3, . . . be the values of V at the

[4] Green writes "shall always give $V = \bar{V}$, for any point within the surface and infinitely near it, whatever may be the assumed value of $\bar{V}$." He also writes (ρ) where we have written ρ.

[1] C. F. Gauss, "Allgemeine Lehrsätze in Beziehung auf die im umgekehrten Verhältnisse des Quadrats der Entfernung . . . ," *Werke*, V, 191–242 [221–226].

$$M_1 v_1 + M_2 v_2 + M_3 v_3 + \cdots = m_1 V_1 + m_2 V_2 + m_3 V_3 + \cdots,$$

which is also expressed by $\sum Mv = \sum mV$, if M [is a variable] representing every mass of the first, m one representing every mass of the second system. Indeed, $\sum Mv$ as well as $\sum mV$ is just the sum ["Aggregat"] of all combinations $M_i m_j / r_{ij}$, where r_{ij} signifies the mutual distance of the points where the corresponding masses M_i and m_j are located.

If the masses of one or both of the systems are not concentrated at discrete points but are distributed continuously on lines, surfaces, or solids, then the above equation remains valid if, in place of the sum, the corresponding integral is substituted.

For example, if the second system of masses is distributed on a surface in such a way that the mass kds lies on the surface element ds, then $Mv = \int kV\, ds$, or if in the first system the surface element ds similarly contains the mass KdS, then $\int Kv\, dS = \int kV\, ds$. It is important to remark in connection with the latter case that this equation remains valid if the two surfaces coincide; for brevity, we shall here only indicate the high points of the way in which this extension of the result can be justified rigorously.[2] For it is not difficult to show that these two integrals, considered with regard to the same surface, can be obtained as limiting values of integrals referring to two separate ["getrennte"] surfaces, by letting the distance between these decrease indefinitely; for this purpose one only needs to consider the two surfaces as congruent ["gleich"] and parallel. This argument is immediately obvious only when the given surface is so constructed that all its normals form acute angles with one straight line. A surface where this condition fails (as it does for any closed surface) will first have to be divided into two or more parts which separately satisfy this condition; thereby it becomes easy to reduce this case to the preceding one.

20.

If we apply the theorem of the preceding section to the case where the second system of masses is distributed over a spherical surface of radius R with uniform density $k = 1$, then the resulting potential v is a constant, namely $4\pi R$, in the interior of the sphere. At each point outside of the sphere, at a distance r from its center, $v = 4\pi R^2/r$, or is equal to the potential at the center of a mass of density $4\pi R^2$ at every point; on the surface of the sphere the two values of v coincide. Hence if the first system of masses is entirely inside the sphere, the $\sum Mv$ is equal to $4\pi R$ times the total mass of this system; but if the system of masses is entirely outside the sphere, then $\sum Mv$ is equal to $4\pi R^2$ times the potential of this mass in the center of the sphere; finally, if the first system of masses is distributed continuously on the surface of the sphere, then both expressions are equally valid for $\int Kv\, dS$. There follows the

THEOREM. If V signifies the potential of a mass distributed over a spherical surface ds of radius R, then, integrating over the entire spherical surface,

$$\int V\, ds = 4\pi (RM^0 + R^2 V^0);$$

where M^0 designates the total mass inside the sphere, V^0 the potential at the center

[2] The standard of rigor here is comparable with that used by Cauchy, Jacobi, and Riemann.

of the sphere due to the external mass, and the masses which happen to be continuously distributed over the surface of the sphere are arbitrarily assigned to either the inner or the outer masses.

21.

THEOREM. The potential V of masses lying outside a connected domain cannot at the same time have one constant value in one part of this domain and a different constant value in another part of it.[3]

Proof. Let us assume that the potential is equal to a constant a at every point of the domain A, and is (algebraically) larger than a at every point of another domain B bordering on A but containing no mass. Construct a sphere, partly contained in B, while the rest and its center are in A; such a construction is always possible. Now if R is the radius of this sphere, and ds an infinitesimal element of its surface, then, according to the theorem of the preceding section, $\int V\,ds = 4\pi R^2 a$, and $\int (V-a)\,ds = 0$, which is impossible since for the part of the surface which lies in A, $V-a=0$, and for the remaining part, by hypothesis, $V-a \neq 0$.

One similarly sees that it is impossible for v to be smaller than a in all points of a domain bordering on A.

Obviously at least one of these cases would have to occur, however, if our theorem were false.

This theorem has the following corollaries:

I. If the region containing the masses surrounds a domain without mass, and the potential is constant in some neighborhood of this domain, then it is constant throughout the domain.

II. If the potential of the masses lying in a bounded ["endlichen"] region has a constant value in any part of external space, then the potential is constant throughout the unbounded external space.

At the same time, one easily sees that in the second case the constant value of the potential must be 0. For if one denotes by M the total mass if they all have the same sign, or in the opposite case the total positive or negative mass, whichever is greater, then the absolute value of the potential is less than M/r at any point whose distance from the nearest element of mass is r, a fraction which can obviously become smaller than any given [positive] quantity in external space.

22.

THEOREM. If ds is the element of a surface bounding a finite connected domain, P the force which arbitrarily distributed masses in ds exert in the direction normal to the surface, a force directed inward or outward being considered positive according as the attracting or the repelling masses are considered to be positive, then, extending the integral over the entire surface,

$$\int P\,ds = 4\pi M + 2\pi M',$$

where M represents the total mass inside the surface, and M' that distributed on it.[4]

Proof. If $U d\mu$ denotes that part of P which results from the mass element $d\mu$, r the distance of the element $d\mu$ from ds, and u the angle which the inward directed

[3] This is clearly a special case of the Principle of the Uniqueness of Harmonic Continuation.
[4] This is Gauss's Integral Theorem, a very powerful tool of potential theory.

normal makes with r in ds, then $U = (\cos u)/r^2$. But, by a lemma proved in §6 of the *Theoria attractionis sphaeroidicorum ellipticorum* with regard to each definite $d\mu$, $\int (\cos u)/r^2\, ds = 0$, 2π, or 4π, depending on whether $d\mu$ is outside the domain bounded by the surface, on the boundary itself, or within that domain. Since $\int P ds$ equals the sum of all the $d \int U ds$, our theorem follows immediately.

It must be remarked that the lemma used here requires a modification from the form in which it is stated in the reference cited above. For r represents the distance of a *given point* from the element ds, and in the case where this point lies on the surface itself, the formula $\int (\cos u)\, ds/r^2 = 2\pi$ holds only as long as the curvature of this bounding surface is continuous. [This is not so], however, if the point lies on an edge or vertex; then, instead of 2π, one must use the solid angle cut out by the totality of straight lines tangent to the surface which emanate from there. Since such exceptional cases concern only lines or points, that is, not *parts* of the surface but only boundaries separating parts, this obviously has no effect on the application made here of the lemma.

23.

We draw a normal through every point of the surface and let p denote the distance of an arbitrary point of this normal from the initial point placed on the surface itself, considered positive on the inside of the surface. The potential V of the masses can be considered as a function of p and two other variables, which somehow distinguish the individual points of the surface from each other. The same is true of the partial derivative $\partial V/\partial p$; however, its value will be considered here only for points [falling] on the surface itself, or for $p = 0$. Since this is completely equivalent with P, if ... no mass is distributed on the surface itself, then,

$$\iint \frac{\partial V}{\partial p}\, ds = 4\pi M.$$

In the case, however, where the whole mass is distributed only on the surface itself, so that the element ds has the mass $k ds$, $\partial V/\partial p$ and P no longer remain equivalent; $\partial V/\partial p$ has two different values, namely $P - 2\pi k$ and $P + 2\pi k$, depending on whether p is to be considered as positive or as negative. Since now $\iint k ds$ obviously will be equal to all of the mass M' distributed over the surface M, and, by the theorem of the preceding section, $\iint P ds = 2\pi M'$, one has

$$\iint \frac{\partial V}{\partial p}\, ds = 0, \qquad \text{or} \qquad \iint \frac{\partial V}{\partial p} \cdot ds = 4\pi M',$$

depending on whether $\partial V/\partial p$ refers to the value on the inside or the outside of the surface, respectively. Hence, in the first case the integral $\int (\partial V/\partial p)\, ds$ is treated exactly as though the mass M' belonged to the outer space, in the second, as though it belonged to the inner space.

Therefore, for arbitrarily distributed masses, the equation $\int (\partial V/\partial p)\, ds = 4\pi M$ is generally valid in the sense that M represents the mass contained in the inner domain, it being understood that, if there also are continuously distributed masses on the surface itself, these are added to the inner ones, or are excluded from them; depending on whether one has chosen the value corresponding to the outward or the inward normal derivative as $\partial V/\partial p$.

Accordingly, if no masses at all lie in the interior of the domain, then at least when $\partial V/\partial p$ is understood to be the value corresponding to thei nterior, $\int (\partial V/\partial p)\, ds = 0$.