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Theta-Pinch Description from Classical Electrodynamics

E. A. Witalis

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Editorial

This issue of the *International Journal of Fusion Energy* focuses on the problem of macroscopic motion in a plasma and its theoretical explanation. Our contributors reflect two contrasting approaches to the problem.

Dr. Pouquet, a member of the respected French group working on fluid and plasma turbulence at the Observatoire de Nice, presents a review of their work on plasma turbulence from a statistical and microscopic standpoint. This approach to the problem has yielded some fruitful insights into the general nature of and basic tendency for highly interacting continua, like plasmas, to form macroscopic structures out of both field variables (fluid flow and magnetic field).

In the second article Dr. Witalis, of the Defense Research Institute in Sweden, attacks the problem of large-scale rotational motion from the standpoint of macroscopic equations of motion. The approach he elaborates in his article has found application in other devices, like the tokamak, that also demonstrate the very general tendency for the generation of large structure out of small in almost every plasma regime.

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Fully Developed Magnetohydrodynamic Turbulence:
Numerical Simulation and Closure Techniques

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Abstract This paper reviews recent results obtained in magnetohydrodynamic (MHD) turbulence at high kinetic and magnetic Reynolds numbers for isotropic, homogeneous, and incompressible flows. The results are based on the extension to MHD of the two-point closures and direct numerical simulations of the primitive equations developed for hydrodynamical turbulence. Closures are particularly useful to investigate questions such as the directions of cascades (which in MHD can be toward both small and large scales) and to analyze nonlocal effects, i.e., interactions involving widely separated scales. Some of the closure-based predictions can then be tested by direct numerical simulations. Both three- and two-dimensional MHD problems are discussed for magnetic Prandtl numbers that are smaller than or of order unity. Particular attention is paid to the generation of large-scale magnetic excitation under the action of helicity (nonlinear dynamo) in three dimensions or of a negative eddy viscosity in two dimensions. Finally, new possibilities for studying large-scale turbulence with the renormalization group technique are evoked.

INTRODUCTION

Concepts and techniques that have evolved in fully developed hydrodynamical turbulence can be extended to other strongly nonlinear problems. By developed (strong) turbulence is meant a flow for which the Reynolds number R^V measuring the relative intensity of the nonlinear term (advection and pressure) to the linear (dissipative) term, is extremely large ($R^V \rightarrow \infty$), with many modes excited and interacting. With direct numerical simulations, only moderate Reynolds numbers have been achieved, of the order of 40 in three dimensions. Two-point closure techniques, introduced in particular by Kraichnan, allow a study of strong turbulence, with Reynolds numbers of the order of 10^6 being handled in the numerical integration of Markovian closure equations. But closures should be viewed more as a model of the original problem than an approximation in which one controls the error. In the search for a small development parameter, renormalization group techniques[1-3] have recently been introduced to turbulence and certainly look promising for study of large-scale (infrared) properties of the flow.

A key concept of hydrodynamical turbulence is the cascade of energy: away from the energy-containing (large-scale) region and the dissipative (small-scale) region, there exists for fully developed turbulence a range called the *inertial range* in which the energy is simply cascading, step by step, to small scales (large wave numbers). The assumption that the energy spectrum $E(k)$ in the inertial range depends only on the rate at which energy is transferred and on the wave number (locality of transfer), immediately yields[4] $E(k) \sim k^{-5/3}$. Two-point closures can be viewed as a quantitative formulation of the Kolmogorov cascade[4] and have been reviewed extensively.[5-9] In this paper, the extension of closure and numerical techniques to homogeneous, isotropic, and incompressible magnetohydrodynamic (MHD) turbulence is reviewed. The main question we shall try to answer concerns the way in which the energy cascade just described is modified for a conducting fluid. Some of the results, in particular those concerning the nonlinear dynamo, have already been reviewed elsewhere.[10, 11]

The MHD equations for an incompressible conducting fluid read

$$\partial \mathbf{v}/\partial t + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \nu\nabla^2\mathbf{v} + (\mathbf{b} \cdot \nabla)\mathbf{b} + \mathbf{f}^V, \qquad (1)$$

$$\partial \mathbf{b}/\partial t + (\mathbf{v} \cdot \nabla)\mathbf{b} = (\mathbf{b} \cdot \nabla)\mathbf{v} + \lambda\nabla^2\mathbf{b} + \mathbf{f}^M, \qquad (2)$$

$$\nabla \cdot \mathbf{v} = 0; \quad \nabla \cdot \mathbf{b} = 0, \tag{3}$$

where $\mathbf{v}$ is the velocity field, $\mathbf{b}$ the magnetic field, ν the viscosity, and λ the magnetic diffusivity; f^V and f^M represent source terms that may be included in the equations. If U_0 is a typical large-scale velocity (at scale L_0), the kinetic and magnetic Reynolds numbers are, respectively,

$$R^V = U_0 L_0 / \nu, \qquad R^M = U_0 L_0 / \lambda. \tag{4}$$

The quadratic spectra that can be introduced in homogeneous isotropic MHD turbulence are the kinetic energy $E^V(k)$, the magnetic energy $E^M(k)$, and the cross-correlation $E^C(k)$ spectra, with

$$\int_0^\infty E^V(k)\, dk = \langle v^2 \rangle; \quad \int_0^\infty E^M(k)\, dk = \langle b^2 \rangle, \tag{5}$$

$$\int_0^\infty E^C(k)\, dk = \langle \mathbf{v} \cdot \mathbf{b} \rangle. \tag{6}$$

If $\langle \mathbf{v} \cdot \mathbf{b} \rangle = 0$ at $t = 0$, it remains so at all times. When the flow is not mirror symmetric, one should include in the various covariance tensors an antisymmetric part[12] and correspondingly introduce the kinetic helicity $H^V(k)$ and magnetic helicity $H^M(k)$ spectra with

$$\int_0^\infty H^V(k)\, dk = \langle \mathbf{v} \cdot \text{curl}\, \mathbf{v} \rangle, \quad \int_0^\infty H^M(k)\, dk = \langle \mathbf{a} \cdot \mathbf{b} \rangle, \tag{7}$$

where $\mathbf{a}$ is the vector potential and $\mathbf{b} = \text{curl}\, \mathbf{a}$ (for details, see Ref. 13).

Experiments

Conducting fluids available in the laboratory have a low magnetic Prandtl number:

$$P_r^M \equiv \nu / \lambda \tag{8}$$

(for example, $P_r^M \sim 10^{-7}$ for mercury at room temperature). Moreover,

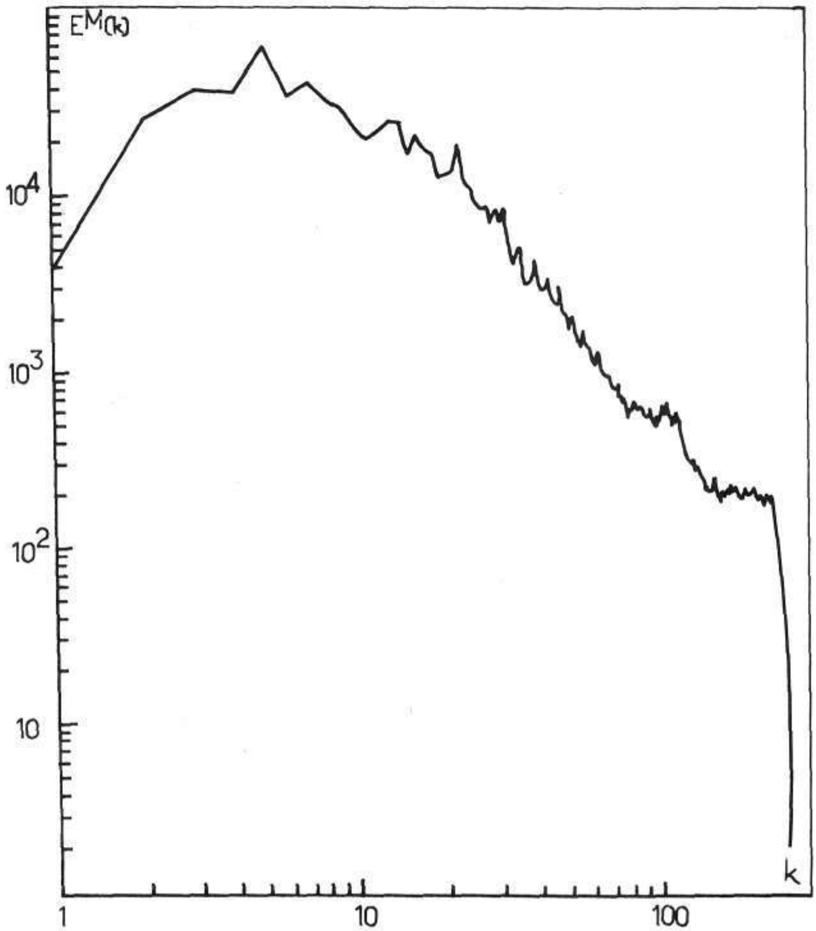

Figure 1. Magnetic energy spectrum obtained by the author from observations of the solar photosphere at the center of the disk at Kitt Peak National Observatory with J. Haryey and W. Livingston (September 1973).

in a typical experiment, the magnetic Reynolds number is smaller than unity so that, for most scales, the induction equation reduces to its (Joule) dissipative part. Experiments of interest for strong MHD turbulence with $R^M \gg 1$ are huge, difficult, and expensive.[14] On the other hand, large magnetic Reynolds numbers are found in cosmic objects: planets and stars are often observed to have an appreciable large-scale magnetic field. In the photospheric field of the sun, many scales of motions can be resolved with present-day Babcock magnetographs, and isotropic spectra can be computed. Figure 1 shows such a magnetic field spectrum taken at the center of the sun at Kitt Peak National Observatory, with a 512-channel magnetograph (in collaboration with J. Harvey and W. Livingston). A review of small-scale solar magnetic fields is given in Refs. 15 and 16. Kinetic and magnetic energy spectra in the solar wind can probably also be obtained from satellite data.[17] Finally, in cosmic rays, the distribution of energy among wave numbers seems to obey a power law[18] with a spectral index γ in the range $0 < \gamma < 3$. The question arises whether such power laws are of a universal nature and whether they can be linked to a turbulent origin.

Numerics

Another source of "experimental data" widely used today is given by numerical calculations, thanks to the versatility of computers. In developed hydrodynamical turbulence, the ratio of energy-containing scales to dissipative scales varies approximately as the 3/4 power of the kinetic Reynolds number. For a three-dimensional simulation of turbulence, the amount of needed storage rapidly exceeds the capacity of present-day computers. To study a flow at the highest Reynolds number possible, one restricts oneself to the incompressible, isotropic, and homogeneous case for which periodic boundary conditions are appropriate. Spectral methods seem best suited in that case.[19] The spectral method of integration consists simply in writing the original equations in Fourier space, and calculating the convolution through a double three-dimensional (3D) or two-dimensional (2D) fast Fourier transform. One keeps only a finite number of modes; initial conditions are generally Gaussian, with a prescribed energy spectrum. Averages are taken over spherical shells of equal width. Aliasing errors, arising from spurious interactions with high out-of-core wave numbers, can be removed,[20] but are found to be negligible both in hydrodynamical and in MHD calculations performed so far.

The main difficulty of such a calculation is in properly handling the huge amount of data: for the 32^3-point MHD code,[21] the total storage needed was of the order of 2×10^6 words (of 64 bits) so that the code constantly used peripheral storage.[22] Bigger codes may become inoperative with the present method because too much machine time is idle while data are transferred from disks to the central core where they are actually processed. For a review of spectral calculations of Navier-Stokes turbulence, see, for example, Ref. 23. Direct numerical simulations of 2D MHD have been performed by Tappert and Hardin,[24] Fyfe, Montgomery, and Joyce,[25] and Orszag and Tang.[26] A spectral method was also used recently[27] in the study of two-dimensional Langmuir collapse in type III solar radio bursts.

Theories

Various theoretical tools are available in the study of turbulence. Mathematical studies stand alone, as their results are definitive but scanty.[9] The central problem remains to demonstrate the possibility of obtaining singularities in a finite time in the Euler (inviscid Navier-Stokes) 3D equation. The results so far state that the solution remains regular up to a finite time depending on initial conditions. This result was extended to MHD by Sulem.[28] For a viscous fluid, global regularity in time of the 3D Navier-Stokes equations (and MHD) has not yet been shown; however, Scheffer[29] has proved that singularities, if any, are very small in the sense of the Hausdorff dimension: their support has a time dimension smaller than 1/2 and a space dimension smaller than unity, making such singularities unobservable by present techniques measuring velocities along a line. It would be interesting to extend Scheffer's results to MHD. This situation is to be opposed to the one in Langmuir collapse. Glassey[30] has shown that for the cubic nonlinear Schrödinger equation, global regularity in time obtains in dimension one ($d = 1$) for which soliton solutions are known, but a blow-up of the solution obtains at a finite time for $d \geqslant 2$.

Quasilinear theory for weak plasma turbulence has proved very fruitful.[31,18] As the amplitude of the wave grows, one progressively enters the realm of strong turbulence. A fascinating breakthrough has been obtained in the study of nonlinear equations with the occurrence of solitons.[32,33] They have blossomed in many fields of nonlinear physics, but not all nonlinear equations can be reduced to a soliton-type equation like the Korteweg de Vries equation! Two approaches to the strongly nonlinear problem can be envisaged: approximations (expansion in a small parameter) and modelization (one-dimensional models or more generally, two-point closure techniques). In strong turbulence, there is no known development parameter at present, except time.

Approximation methods, however, have gained new vigor with the introduction by Forster and co-workers[2] of the renormalization group technique to turbulence (see concluding section of this paper). We shall be concerned mainly with the two-point closure techniques described in the next section. Results for three-dimensional magnetohydrodynamic turbulence obtained with closures and with direct numerical simulations are given in the section entitled "Results in 3D MHD". The magnetic Prandtl number (8) is equal to unity in all cases except for the subsection entitled "Variation with Magnetic Prandtl Number." The main results concern the Alfvén effect (quasiequipartition of kinetic and magnetic energy in the small scales) and the dynamo effect (growth of large-scale magnetic field when the flow contains helicity). The 2D MHD case is dealt with in the next section; it is shown in particular that 2D MHD may have strong similarities in the small scales to 3D turbulence and could possibly be taken as a model of intermittency. In the last section of the paper, difficulties of current turbulence research are evoked. Perspectives of future research are also given, in particular the possibility of calculating large-scale properties of turbulence with the renormalization group technique.

CLOSURE TECHNIQUES

Method

For the purpose of exposition, let us write the following symbolic equation:

$$dX/dt = XX, \tag{9}$$

which stresses the main source of difficulty in the Navier-Stokes (NS) and MHD problems, namely, the nonlinearity. In MHD, X would represent the velocity and the magnetic field. We assume isotropy, homogeneity, and incompressibility for the X field. The linear (dissipative) term is omitted since it poses no particular problem. The derivation can also be performed for more general cases. The stochasticity assumed in Eq. (9), through random initial conditions or random forces, can be seen as arising from the nonlinearity: two initial conditions identical except in the small scales become uncorrelated in a time of the order of the turnover time. This has important consequences in meteorology: the lack of weather data on scales smaller than the average distance between meteorological stations prevents weather prediction (even with an idealized accurate model) to be made for time longer than a predictability time of the order of two weeks[34] (see also Ref. 79).

In a statistical theory of turbulence, one is interested in the determination of the various moments of the X field. In a homogeneous and isotropic situation, we can assume $\langle X \rangle = 0$ without loss of generality: a uniform velocity field can be eliminated in a Galilean transformation, whereas a uniform magnetic field has nontrivial effects on small scales (Alfvén waves) but would break the isotropy assumption. From (9), the equations for the second- and third-order moments are

$$d\langle XX \rangle / dt = \langle XXX \rangle \tag{10}$$

and

$$d\langle XXX \rangle / dt = \langle XXXX \rangle . \tag{11}$$

The fourth-order cumulant $\langle XXXX \rangle_c$ is defined as

$$\langle XXXX \rangle_c \equiv \langle XXXX \rangle - \langle XX \rangle \langle XX \rangle , \tag{12}$$

where indices and sums are omitted.

Each new equation in the hierarchy of moments brings a new unknown, and one needs an additional physical assumption to close the stochastic problem. For a field with Gaussian statistics, one has

$$<XXXX>_c = 0. \tag{13}$$

Take Gaussian initial conditions. The quasinormal (QN) approximation consists in assuming that Eq. (13) holds for all times and in Markovianizing the solution of Eq. (11) thus obtained. This approximation is exact for small times to order t^3 as can be seen by making a Taylor expansion in time of Eq. (9) (see, for example, Ref. 35). This approximation yields negative energy spectra.[36, 37] Indeed, the zero fourth-order cumulant assumption deprives the triple correlation (transfer) function appearing in Eq. (10) of a relaxation mechanism.[8] One may simulate this relaxation by expressing the unknown fourth-order cumulant in terms of a *linear* relaxation of third-order moments:

$$d <XXX>/dt = <XX> <XX> - \mu(<XX>) <XXX>, \tag{14}$$

where μ is a relaxation operator depending only on second-order moments and whose expression remains to be determined.

Eddy-Damped Quasinormal Markovian Approximation (EDQNM)

With the relaxation of triple correlations introduced in Eq. (14) and once Markovianization in solving Eq. (11) has been performed, the hierarchy of moments is closed at the level of the energy spectra. Various closures can be obtained by different choices of the eddy-relaxation operator μ. For example, it may be determined by phenomenological arguments: The two local characteristic times one can write in hydrodynamical turbulence are the viscous damping time $(\nu k^2)^{-1}$ and the eddy turnover time $(k^3E(k))^{-1/2}$ for an eddy of scale $l = k^{-1}$ and energy $v_l^2 \sim kE(k)$. By taking

$$\mu(k) = \nu k^2 + \left[k^3E(k)\right]^{1/2}, \tag{15}$$

one obtains the eddy-damped quasinormal Markovian (EDQNM) approximation,[38] which leads to a $-5/3$ Kolmogorov range. In MHD, another characteristic time can be constructed, the Alfvén time $(kb_0)^{-1}$, where b_0 is the rms large-scale magnetic field. For large enough wave numbers, the Alfvén time will be shorter than the eddy turnover time and nonlinear transfer will be dominated by Alfvén-wave exchanges between kinetic and magnetic energy. This is thought to modify the Kolmogorov spectrum[39] from a $-5/3$ to a $-3/2$ law. The following choice for the relaxation operator[40]:

$$\mu(k) = (\nu + \lambda)k^2 + C_1[k^3(E^V(k) + E^M(k))]^{1/2} + C_2\left[\int_0^k E^M(p)\, dp\right]^{1/2} \tag{16}$$

indeed gives a kinetic and magnetic energy inertial range in $-3/2$ (see 3D MHD results below). As the problem treated grows in complexity, the degree of arbitrariness in the choice of the relaxation operator of triple correlations μ grows also. It must be stressed, however, that, from a qualitative point of view (indication of the direction of the cascade, for example), the precise choice of μ may not matter so much. Even the choice $\mu =$ const yields interesting information,[41] and is easier to handle for mathematical studies.[42]

Remark

The relaxation operator μ can also be determined through the introduction of an auxiliary problem,[43] namely, the advection of a test field by a turbulent velocity field—hence the name of this model, the test field model (TFM) introduced by Kraichnan.[44] An MHD version of the TFM has not been written to date but would probably yield essentially the same results as the EDQNM: one can show that the latter model can be recovered by a simplification of the former for hydrodynamical turbulence.[45]

Non-Markovian two-point closures have been introduced in hydrodynamical turbulence, in particular the direct interaction approximation (DIA) of Kraichnan.[46] The DIA corresponds to a partial resummation of diagrams appearing in a formal expansion of the Navier-Stokes equation. In the DIA, the nonlinear vertex is not renormalized.[47-49, 35] In the case of nonhelical MHD, the expansion was formally carried out by Lee[50]; it reproduced, in its simplest form, results obtained by Chandrasekhar. However, the DIA treats improperly the advection of small-scale eddies by large scales.[51] The TFM was, in fact, introduced to cure this defect.

Nonlocal Expansions

The EDQNM equations, with $\langle \mathbf{v} \cdot \mathbf{b} \rangle \equiv 0$, are given in Ref. 40 for three-dimensional (3D) helical MHD and in Ref. 52 for 2D MHD. The nonhelical 3D MHD closure equations are also found in Refs. 53 and 54. One obtains a set of coupled integrodifferential equations for the kinetic and magnetic energy (and helicity in 3D) spectra. In the nonhelical case, these equations are of the form (omitting dissipative and forcing terms)

$$\partial E^{\alpha}(k)/\partial t = T^{\alpha}(k, p, q) \tag{17}$$
$$= \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} dp\, dq\, [A^{\alpha\beta\gamma}(k, p, q)E^{\beta}(p)E^{\gamma}(q) - B^{\alpha\beta}(k, p, q)E^{\beta}(q)E^{\alpha}(k)],$$

where in the expression for the nonlinear transfer $T^{\alpha}(k, p, q)$, the superscripts $\alpha, \beta, \gamma \equiv V$ or M denote a kinetic or magnetic energy spectrum and where the triad interaction with $\mathbf{k} = \mathbf{p} + \mathbf{q}$ stems from the convolution of the nonlinear term of Eqs. (1) and (2) in Fourier space. The first term on the r.h.s. of (17) is an emission term (the A coefficient can be shown to be positive), and the second term, linear in $E^{\alpha}(k)$, is called an absorption term.

The transfer is said to be *local* if most of the nonlinear interactions at mode k come from wave numbers of roughly the same size [as depicted in Figure 2(a)]. This is the case for 3D hydrodynamical turbulence in the inertial range.[55] However, in MHD, nonlocal effects corresponding to the interaction of modes of widely separated scales [Figure 2(b)] contribute in an important way to the nonlinear transfer: a small-scale hydrodynamic turbulence stretches the lines of a large-scale magnetic field, thereby producing small-scale magnetic energy;

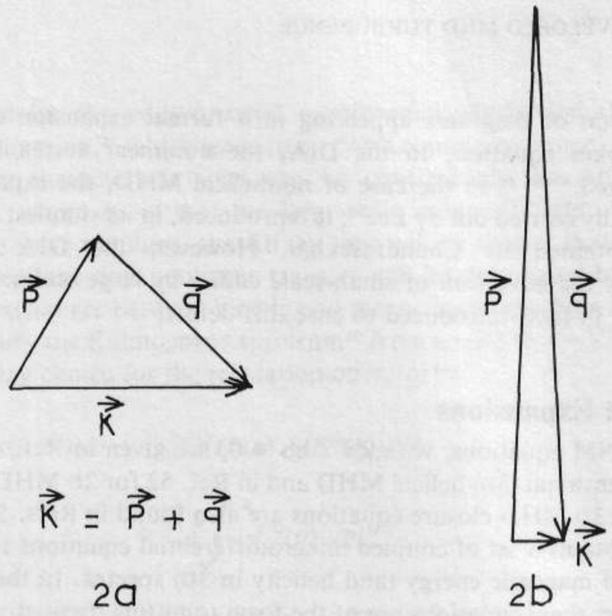

Figure 2. Interacting triad in turbulence: (a) local interaction; (b) nonlocal interaction involving modes of widely separated scales.

and if the small-scale turbulence is helical, it also twists the lines of the magnetic field, thereby reinforcing the large-scale magnetic field (cyclonic event[56]). Nonlocal effects can be displayed on closure equations by making a Taylor expansion in the ratio of small-to-large wave numbers [k/p in the case of Figure 2(b)] for the various transfer terms.[40] For example, in 3D hydrodynamical turbulence, the equation for the kinetic energy spectrum, when $k \ll p \sim q$ and when only the absorption term proportional to $E^V(k)$ is taken into account, reduces to

$$\partial E^V(k)/\partial t = -2\,(\nu + \nu_{\text{turb}})\, k^2 E^V(k). \tag{18}$$

The coefficient ν_{turb}, proportional to the energy contained in the small scales, is called a turbulent eddy viscosity. One recovers here the familiar idea that turbulence, by enhancing the transfer of energy from large to small scales, enhances the transport coefficients, and hence the dissipation. This effect is reminiscent of the mass (linear term) renormalization of quantum field theory. The eddy viscosity is found to be negative in some cases, corresponding to a destabilization of the large scales by small-scale turbulence.[52, 57-59]

The numerical integration of the EDQNM closure equations presents no particular difficulty. Because of (presumably) the existence of power-law inertial ranges, one can take an exponential discretization in wave number. Nonlocal effects may be included separately.[60, 40]

RESULTS IN 3D MHD

Absolute Equilibrium Ensemble: A First Hint

A first investigation of the properties of nonlinear MHD consists in studying the nondissipative case. When $\nu = \lambda = 0$, several invariants of the motion are obtained, some of which are conserved in detail, i.e., conserved for *each* interacting triad (k, p, q). These invariants, which are quadratic,[61] will therefore survive truncation in Fourier space when only a finite number of modes is kept. It can be shown that the Navier-Stokes (and MHD) system will then relax to a Gaussian equilibrium distribution $P(X) \sim \exp(-\alpha I)$, where I is a linear combination of the quadratic invariants of the problem (listed in Table 1 for MHD).

The interest of studying nondissipative equilibrium lies in the possibility of predicting from them the direction of transfer.[62, 9] For 3D MHD with nonzero magnetic helicity, one obtains magnetic equilibrium spectra that peak at low wave numbers.[13] From this, one can

Table 1. Quadratic invariants of inviscid ($\nu = \lambda = 0$) magnetohydrodynamics.

Quantity	2D MHD	3D MHD
Total energy	$\langle v^2\rangle + \langle b^2\rangle$	$\langle v^2\rangle + \langle b^2\rangle$
Cross correlation	$\langle \mathbf{v} \cdot \mathbf{b}\rangle$	$\langle \mathbf{v} \cdot \mathbf{b}\rangle$
Magnetic helicity	—	$\langle \mathbf{a} \cdot \mathbf{b}\rangle$ (where $\mathbf{b} = \text{curl } \mathbf{a}$)
Squared magnetic potential	$\langle a^2\rangle$	—

infer that, in the dissipative case, magnetic helicity will transfer to the large scales. The 2D MHD case was treated by Fyfe and Montgomery,[63] who similarly postulated an inverse cascade of squared magnetic potential to the large scales (see 2D results below).

Small-Scale MHD: The Alfvén Effect

A uniform magnetic field couples small-scale kinetic and magnetic fields through Alfvén waves. In the isotropic case, with a wide range of scales, a similar effect is found, the large-scale magnetic field being seen by the small scales as uniform. A nonlocal expansion of the EDQNM equations shows that the relative energy $E^R(k)$ and helicity $H^R(k)$ at wave number k defined by

$$E^R(k) = E^V(k) - E^M(k) \tag{19}$$

and

$$H^R(k) = H^V(k) - k^2 H^M(k) \tag{20}$$

relax to zero in a time $(kb_0)^{-1}$ where b_0 is the rms large-scale magnetic field.

A numerical integration of the EDQNM equations confirms that result, but shows a slight *excess* of magnetic energy in the small scales probably owing to the fact that velocity gradients directly stretch magnetic field lines but act on the velocity field only through the Lorentz force. The total (kinetic plus magnetic) energy cascades to the small scales and with the choice (16) for the relaxation of triple correlations, a $-3/2$ range obtains. A measurement of kinetic and magnetic energy spectra in astrophysical situations (with high Reynolds number) would be of great interest.

Large-Scale 3D MHD: The Helicity Effect

In 3D MHD, as for 3D hydrodynamical turbulence, small scales act as a drain of the energy contained in the large scales. A nonlocal expansion of the absorption terms of the EDQNM equations gives results similar to Eq. (18) for the kinetic and magnetic energy spectra with positive kinetic and magnetic eddy viscosities.[40] In the presence of helicity, however, another very important nonlocal effect of the small scales upon the large scales is observed, the α (or helicity) effect. It is found that small-scale residual helicity destabilizes magnetic energy and helicity in the large scale K^{-1} in the following way (with eddy viscosities omitted):

$$\partial E^V(K)/\partial t = \partial H^V(K)/\partial t = 0, \tag{21}$$

$$\partial E^M(K)/\partial t = \alpha^R K^2 H^M(K), \tag{22}$$

$$\partial H^M(K)/\partial t = \alpha^R E^M(K), \tag{23}$$

$$\alpha^R = \int_> \frac{dq}{2\mu(q)} \, [H^V(q) - q^2 H^M(q)], \tag{24}$$

where $\int_>$ stands for integration on small-scale turbulence. Expressions (21) through (24) are obtained from a nonlocal expansion of the EDQNM equations.[40] This destabilization of large-scale magnetic excitation is linked to the inverse cascade of magnetic helicity postulated on the basis of statistical equilibrium.[13] A numerical integration of the EDQNM equations with a source term of magnetic energy and helicity shows that magnetic excitation is transferred to larger and larger scales as time elapses (Figure 3), whereas the energy cascades to the small scales.

This inverse cascade of magnetic helicity has interesting consequences for the problem of generation of large-scale magnetic fields. It was already known from linear analysis (kinematic dynamo) that the kinetic helicity contained in the small scales destabilizes large-scale magnetic fields.[64, 65] The EDQNM closure allows the study of the full nonlinear problem.

Consider an initial magnetic seed field of intensity b_0 imbedded in a strong kinetic turbulence with kinetic energy and helicity forcing in a

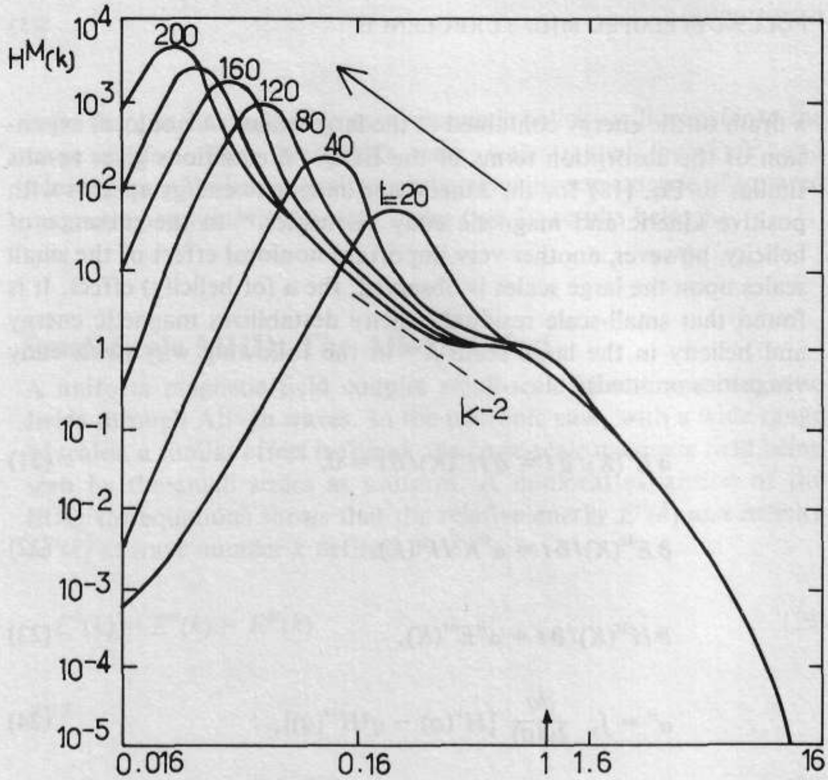

Figure 3. The inverse cascade of magnetic helicity (taken from Ref. 40). Kinetic and magnetic energy and magnetic helicity are injected at $k = 1$.

narrow band of wave numbers but with no magnetic forcing ($f^M \equiv 0$ in Eq. 2). The kinetic helicity could, for example, stem from rotation and stratification. We already have all the necessary ingredients to understand how the dynamo actually works. The kinetic and magnetic energy [$E^V(k)$ and $E^M(k)$] and the kinetic helicity [$H^V(k)$] all cascade to small scales through nonlinear interactions. Through the Alfvén effect (stretching of magnetic field lines), magnetic helicity, of which there was none initially, is produced in the small scales with $H^M(k) \sim H^V(k)/k^2$ [see Eq. (20)]. In the large scales, magnetic helicity of the opposite sign appears (H^M is a pseudoscalar) because of the dynamical constraints of conservation of total magnetic helicity. Furthermore, the dissipation of magnetic helicity in the small scales caused by Joule

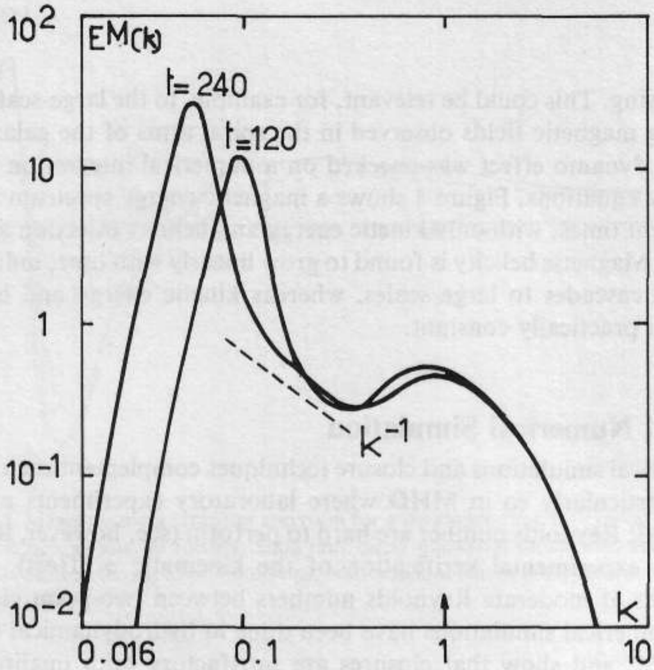

Figure 4. Magnetic energy spectrum in the turbulent dynamo (taken from Ref. 40). Injection of only kinetic energy and helicity at $k = 1$. Notice the buildup of large-scale magnetic energy.

damping acts as a *source* of magnetic helicity of opposite sign in the large scales.

This large-scale magnetic helicity, in turn, cascades toward larger scales and carries along magnetic energy. By this circuitous process, magnetic excitation is produced in scales larger than the kinetic injection scale. The EDQNM closure indicates that, in the nonlinear regime, the motor of this instability is proportional to the *relative helicity* contained in the small scales. For an initial weak magnetic energy, the helicity effect reduces to its kinetic part, in agreement with previous kinematic studies.[64, 65] The invariance of magnetic helicity (for $\nu = \lambda = 0$) plays an essential role in this dynamo effect; a phenomenological argument can be given to show the effect of magnetic helicity on the growth of large-scale fields.[40] It must be stressed that in isotropic MHD turbulence, it is the large-scale fluctuating energy that is growing. This could be relevant, for example, to the large-scale fluctuating magnetic fields observed in the spiral arms of the galaxy.[66]

The dynamo effect was checked on a numerical integration of the closure equations. Figure 4 shows a magnetic energy spectrum at two different times, with only kinetic energy and helicity injection around $k = 1$. Magnetic helicity is found to grow linearly with time, indicating that it cascades to large scales, whereas kinetic energy and helicity remain practically constant.

Direct Numerical Simulation

Numerical simulations and closure techniques complement each other, and particularly so in MHD where laboratory experiments at high magnetic Reynolds number are hard to perform (see, however, Ref. 67 for an experimental verification of the kinematic α effect). Comparisons at moderate Reynolds numbers between two-point closures and numerical simulations have been done in hydrodynamical turbulence[68, 69] and show that closures are satisfactory both qualitatively and quantitatively at those Reynolds numbers. It remains to be seen whether such good agreement still holds as the Reynolds number is increased together with the size of available computers.

The time-step scheme used in Ref. 21 to simulate three-dimensional helical MHD turbulence is a leap-frog, with a linear term treated implicitly. On the CDC 7600 of the National Center for Atmospheric Research, the 32^3-point MHD code took roughly six seconds per time step. The Reynolds number achieved was of order 40. In such a case, no inertial range to speak of can be observed, but global properties of the flow (such as the direction of transfer) are given.

The results obtained in decay calculations [$f^V \equiv 0$, $f^M \equiv 0$ in Eqs. (1) and (2)] confirm the theoretical predictions of the closure. In particular, in the small scales, one observes a slight excess of magnetic energy over kinetic energy. Also, magnetic helicity is seen to grow in the large scales of the flow. In Figure 5 is represented the magnetic transfer spectrum $T^M(k)$, defined as

$$\partial E^M(k)/\partial t + \lambda k^2 E^M(k) = T^M(k), \tag{25}$$

at a typical turnover time for a decay calculation (see Ref. 21 for

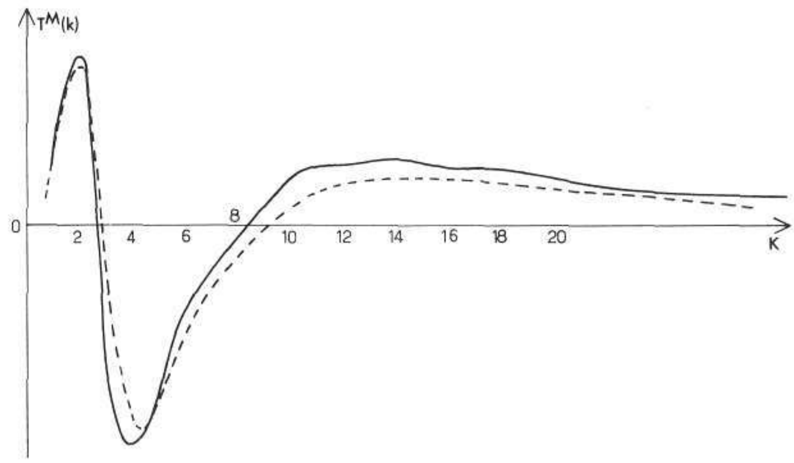

Figure 5. Magnetic energy transfer spectrum for a developed flow with initial maximal magnetic helicity and no forcing. Solid line: direct numerical simulation; dotted line: EDQNM closure for the same conditions. Notice that magnetic energy is transferred to the large scales.

details). The dotted line indicates the closure calculation under the same conditions, and the solid line the numerical simulation. Magnetic energy is removed from wave number $k \sim 5$, where it peaks initially, and is transferred to large scales [positive $T^M(k)$ for $k < 5$]. For comparison, kinetic energy transfer defined in a similar fashion as (25) is plotted in Figure 6, where it is seen to be transferred only to the small scales. At moderate Reynolds number, the EDQNM closure for 3D MHD is in rather good agreement with the direct numerical simulations of the MHD equations, as was also the case for various closures in hydrodynamical turbulence.

The effect of the cross correlation $\langle \mathbf{v} \cdot \mathbf{b} \rangle$ on the existence of inverse transfer of magnetic helicity was checked with numerical simulations. It is in principle tractable by closures but implies very long algebra. As expected from a direct observation of the MHD equations, the dynamics is slowed down, but the magnetic excitation is still seen to be transferred to larger scales.

Variation with Magnetic Prandtl Number

Most conducting fluids in astrophysical objects or in the laboratory do not have a magnetic Prandtl number P_r^M equal to unity. Kraichnan

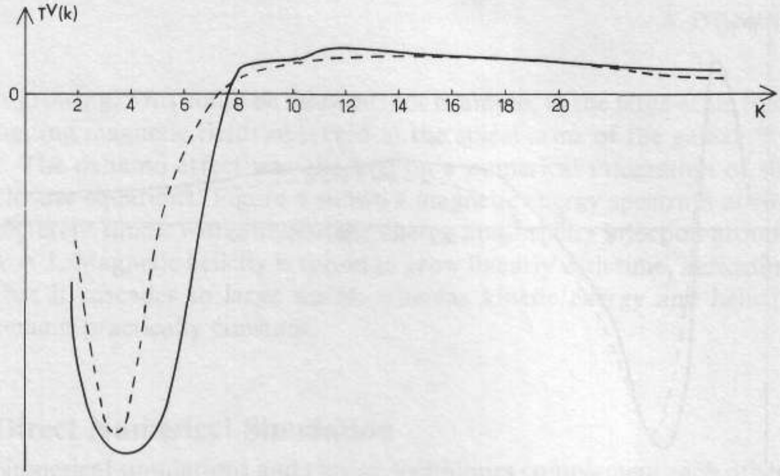

Figure 6. Kinetic energy transfer function. Same conditions as in Figure 5.

and Nagarajan[70] have given a detailed description of possible inertial ranges for $P_r^M \ll 1$. Numerically, a magnetic Prandtl number differing much from unity is difficult to treat because of the wide range of scales needed to describe the turbulence. We simulated numerically a decay problem with initial maximal magnetic helicity, initial kinetic and magnetic energies equal, and a magnetic Prandtl number equal to 0.2 ($R^M = 8$ with $R^V = 40$). Although Joule dissipation is five times greater than viscous dissipation, the magnetic-to-kinetic energy ratio $\psi(t) \equiv \langle b^2(t)\rangle / \langle v^2(t)\rangle$ is still found to grow (with a maximum at 1.4), and inverse transfer of magnetic energy and helicity is still observed for $P_r^M = 0.2$. For a magnetic Reynolds number of the order of 4 ($P_r^M = 0.1$), the energy ratio $\psi(t)$ diminishes at all times, indicating that magnetic field-line stretching is overwhelmed by Joule dissipation at that low magnetic Reynolds number. Figure 7 shows the variation of ψ with magnetic Prandtl number at a time when the turbulence has developed [the total enstrophy defined in Eq. (26) below having reached its maximum].

Among the numerical simulations that are envisaged in the future, one is particularly worth mentioning. It concerns the possible existence of a critical magnetic Reynolds number[71] above which a seed magnetic field may grow spontaneously in strong ($R^V \to \infty$) turbulence with a source term in the equation for the velocity field [$f^V \neq 0$ in Eq. (1)] but with no magnetic forcing [$f^M \equiv 0$ in Eq. (2)]. Calculations

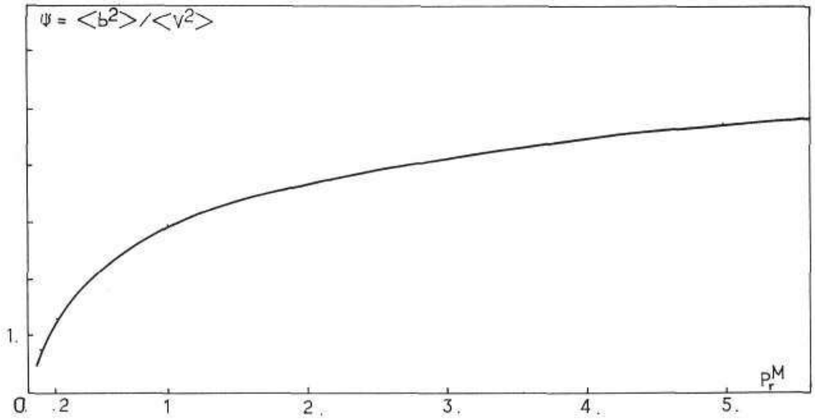

Figure 7. Magnetic-to-kinetic energy ratio as a function of magnetic Prandtl number in a decay calculation. Initially, $\psi = \langle b^2 \rangle / \langle v^2 \rangle$ is unity.

with the EDQNM closure[72] indicate that this may be the case with the critical magnetic Reynolds number, being of order 30 for the nonhelical case. The result could be of some interest in the problem of the building of breeder reactors.[73] As their size is increased, the magnetic Reynolds number in the liquid sodium coolant could cross the critical value leading to the appearance of measurable magnetic fields. Closure calculations[72] indicate that, for a supercritical magnetic Reynolds number, one should expect a reduction in the turbulent kinetic energy, slight in the large scales but becoming increasingly important in the small scales because of Joule dissipation.

Results in 2D MHD

The dynamics of 2D MHD turbulence differs from that of the 3D case because of the presence of an additional quadratic invariant, the squared magnetic potential (see Table 1). Fyfe and Montgomery[63] investigated the nondissipative case and thereof postulated the existence of an inverse cascade of magnetic potential to the large scales. A numerical simulation in a 32 × 32-point cyclic box was performed using a spectral method[25] and corroborated such inverse transfer. This was also verified on closures[52] and was associated with a negative eddy viscosity: small-scale magnetic turbulence destabilizes the large-scale magnetic potential A. The flux of magnetic potential, when averaged over the small-scale turbulence, is found to be in the *same* direction as the gradient of A, yielding to the heat equation with a negative diffusion coefficient.[52]

More recently, attention was focused on the structure of small-scale 2D MHD turbulence because of possible similarities to 3D turbulence. In two dimensions, the Euler equation is known to remain regular for all time.[74] This can be linked to the conservation of kinetic enstrophy

$$\Omega = \langle (\mathrm{curl}\ \mathbf{v})^2 \rangle$$

in the inviscid case. Since this conservation is broken by the Lorentz force in MHD, it was therefore conjectured that nondissipative 2D MHD could blow up in a finite time,[52] just as 3D turbulence is thought to do (this has not yet been proved; see Ref. 9 for a review of the mathematical situation of turbulence). A numerical integration of the EDQNM equations indicates that such a blow-up of total enstrophy is plausible in the limit of infinite Reynolds numbers, and a direct numerical simulation on a 256 × 256 grid[26] confirmed this possibility.

The closure also predicts that this blow-up of total enstrophy Ω:

$$\Omega = \langle (\mathrm{curl}\ \mathbf{v})^2 \rangle + \langle (\mathrm{curl}\ \mathbf{b})^2 \rangle \tag{26}$$

is accompanied by an energy dissipation in the limit of zero viscosity and magnetic diffusivity. The numerical simulation, however, seems to contradict this latter result. If this is corroborated by further numerical simulations, it certainly presents a significant challenge to the closure community, insofar as it is the first time that such a qualitative unexplained discrepancy between numerics and closures has been encountered. The (faster than exponential) growth of total enstrophy Ω is associated to a flow with finer and finer small-scale structures in which dissipation occurs. This is linked to problems of intermittency and geometry of small-scale turbulence on which 2D MHD could possibly shed light (see concluding section).

Conclusions and Perspectives

Many works have been devoted, in recent years, to the understanding of the onset of turbulence from a mathematical point of view[75-77] as well as from experiments[78] and numerics. Particular attention was paid to the appearance of stochasticity in coupled nonlinear ordinary differential equations, for example in connection with strange attractors.[79-83] Such systems contain only a few spatial modes but may display randomness in time. Fully developed turbulence, on the other hand, has spatial as well as temporal randomness.

Two-point closure techniques have proved very useful in the study of strong nonlinear problems: hydrodynamical and MHD turbulence, Vlasov turbulence,[84,85] influence of a strong external magnetic field on homogeneous MHD turbulence,[86] relaxation of anisotropy,[87] chemical turbulence,[88] spin hydrodynamics,[89] barotropic Rossby waves,[90] and strong turbulence in magnetized plasmas.[91] The extension of two-point closures to inhomogeneous turbulence may prove to be untractable (see, however, the one-point clipping approximation in Ref. 92).

In hydrodynamics, two-point closures can be viewed as a quantitative formulation of the phenomenological ideas of Kolmogorov[4] leading to a $-5/3$ inertial range for the energy spectrum. However, experiments for higher-order correlation functions indicate that the Kolmogorov predictions are incorrect.[93] Several modifications to the Kolmogorov law have been proposed.[94,95,35] The question of the correction to the 5/3 law remains open. It has been linked to the intermittent structure of small-scale turbulence,[96,97] with smaller eddies filling less physical space as the cascade proceeds. This problem of intermittency with very fine, strongly concentrated structures separated by wide portions of quiescent flow (spottiness) is not described by the two-point closures and appears as a stumbling block of turbulence theory. More measurements and visualization of the geometry of small-scale turbulence are needed.

Another approach consists in looking at field lines in a numerical experiment. In 3D simulations, the Reynolds number is not large enough to see appreciable effects of intermittency.[98,99] In two dimensions, however, higher Reynolds numbers can be attained; in that respect, 2D MHD might be considered as a prime experimental (numerical) field for problems of intermittency. The problem of small-scale structure of turbulence is linked to the possible loss of regularity of the inviscid 3D Navier-Stokes equation (and probably also of the MHD equations) at a finite time. Such an approach to a singular behavior, as the viscosity and magnetic diffusivity tend to zero, could be studied numerically in 2D MHD, in particular by looking at the stretching and twisting of vortex and current lines by velocity gradients.

A promising tool in the study of the large-scale properties of turbulence seems to be the renormalization group technique (RG) developed in critical phenomena.[100,101] Forster, Nelson, and Stephen[1,2] were the first to introduce the RG to study the infrared ($k \rightarrow 0$) properties of randomly forced, homogeneous, isotropic, and incompressible Navier-Stokes turbulence.[102] In the renormalization group technique, a small expansion parameter linked to the dimension of space is displayed. Fournier[103] showed that it was possible, in turbulence, to calculate in a fixed dimension by expanding in a parameter linked to the spectral exponent of the force autocorrelation. The critical dimension (above which the Navier-Stokes equation reduces to a Langevin equation) is replaced by a critical spectral index.

With this modification, one can perform renormalization group calculations of helical turbulence in three dimensions.[3] For Navier-Stokes turbulence helicity is, at best, marginally relevant (it does not modify the fixed point of nonhelical turbulence). In helical MHD, however, the new linear term (generated by the RG procedure) which appears in the induction equation is proportional to curl **b** and dominates Joule dissipation as $k \rightarrow 0$; this destabilizing term corresponds to the dynamo effect. In nonhelical MHD, preliminary calculations[104] indicate that for a weak magnetic forcing, the renormalized magnetic Prandtl number tends to a universal value in the large scales. As the magnetic forcing is increased and reaches a critical level, a new fixed point appears by which the magnetic Prandtl number tends to infinity and the inertial term becomes negligible. It seems that in that case large-scale motions result from a balance between the Lorentz force and viscous dissipation, leading to a cubic equation for the evolution of the magnetic field. Renormalization group techniques seem promising in the study of large-scale phenomena (turbulence with rotation, turbulence in the presence of a strong uniform magnetic field). It is not clear how they can be extended to the ultraviolet end of the spectrum to study intermittency.

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Theta-Pinch Description from Classical Electrodynamics

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Abstract The large and extensively reported discrepancies between theoretically expected and experimentally observed theta-pinch properties are briefly reviewed. The denial in basic MHD theory of the observed strong plasma mass rotation can be traced back to a too simplified and restrictive plasma single-fluid formulation. Adopting a two-fluid description, a basic equation for ion motion is rewritten so as to express, in a moving frame, conservation of ionic canonical angular momentum. The electronic charge transport in fluid and particle formulations is discussed and expressed quite generally as a magnetization current. The combined ion and electron charge equations yield upon integration a linear differential equation for which the presented solutions relate the magnetic flux within the plasma to the plasma column expansion or compression. Special attention is given to solutions that describe much discussed observations like the separation of the imploding plasma density gradient ahead of or behind the magnetic field gradient, or an azimuthal ion velocity reversal. The "magnetic piston" concept is criticized. Power and energy considerations from classical electromagnetic theory of magnetized media and linear current systems explain the remarkable influence of initial bias field upon the implosion dynamics.

INTRODUCTION

The magnetic compression arrangement known as the fast theta pinch still remains the only controlled magnetic fusion device that readily produces truly thermonuclear plasma conditions. From the well-established classical MHD theory, both in macroscopic and guiding center formulations (proved to yield identical results[1]) its operation principle is simple. Roughly, an increasing axial magnetic field radially compresses and thereby also heats a highly conducting plasma column. However, decades (both in years and number of experiments) have repeatedly proved that the theta-pinch plasma cares very little about classical MHD theory. The predicted concentrated surface currents never show up; the ions are the first to become hot instead of the electrons[2]; although MHD theory denies any torque,[1] the plasma column immediately starts to rotate, and up to such a tremendous speed that even the emission of megaelectron volt fusion neutrons is rotationally shifted[3]; some bias field cannot be expected to change the compression very much but actually will, depending on its direction, change the implosion dynamics drastically[2,4,5]; the latter is often treated theoretically as the action of a magnetic piston or snow-plow[2,4,5]; at higher initial densities, however, this "plow" has been observed to be capable also of dragging![6,7]

In contrast to the fairly well-understood tokamak plasmas for which, however, experimental parameters of today remain about the same as in 1968, there is a steady flow of intriguing results from theta-pinch experiments. For example, Ref. 8 reports strong plasma heating near the compression coil ends; Ref. 9 observes plasma outflow there in conflict with theories; Ref. 10 investigates a remarkable ion-separation mechanism in the plasma column; Ref. 11 reports extreme heating with kiloelectron volt x radiation from field reversal regions; and Ref. 12 stresses the decisive influence of the initial plasma density upon the imploding plasma structure.

Recent theoretical efforts are dominated by *one* of the many radical departures from classically expected behavior, namely, the rapid external field penetration or "diffusion" into the plasma. (See the brilliant discussion in Ref. 13 about the use of the diffusion concept in plasma contexts.) Among such papers on "anomalous resistivity," Ref. 14 is usually referred to. It makes use of very extensive numerical calculations that include turbulence, three linear kinds of instability, and nonlinear instability saturation mechanisms. In conclusion, the authors present a formidable list of neglected effects and still more instability types to be included in future codes, which may yield satisfactory agreement with experiments.

MAGNETICALLY INDUCED PLASMA ROTATION

In 1968 we showed[15] that the two standard MHD equations for mass and charge transport, with, of course, the full convective acceleration term retained for the former and tensor conductivity for the latter, imply that a plasma picks up mechanical angular momentum from the rotational electric field associated with time and, for a moving plasma, space variations of a magnetic field. This result, i.e., the existence of a form of the Einstein-de Haas effect for plasmas also, is in glaring conflict with the generally accepted fluid formulation disproof[16] of electromagnetic origin for the intense rotation observed in theta-pinch plasmas. An impressive number of explanations compatible with (or rather suggested by) the disproof were proposed but rejected by the results from accepted theory[1] and careful experiments[7] that quickly turned up. As a result the question of rotation origin has become so controversial and disliked by plasma journals that it is avoided even in works directly addressed to the structure and consequences[17,18] of the rotation.

To resolve the aforementioned conflict it must be observed that a plasma is *not* a continuum, but a particle system, for which the interaction between particles is *not* of purely central force character. Any absence, resulting from quasineutrality, of net electric torque on any plasma unit volume does *not* imply conservation of mechanical angular momentum. The proper continuum formulation and technique[19] for handling such particle systems was proved,[20] with electron inertia retained, to reproduce our result from the simple MHD equations. The analysis and the results presented in the sequel are actually little more than the application of the outcome of Ref. 20.

THE ION MOTION EQUATIONS

The nonviscous motion for singly ionized ions is described by the equation

$$m_i n_i (d\mathbf{V}_i/dt) = e n_i(\mathbf{E} + \mathbf{V}_i \times \mathbf{B}) - \operatorname{grad} p_i - e n_i(\mathbf{j}/\sigma), \tag{1}$$

where m_i, n_i, e_i, V_i and p_i are the ion mass, density, charge, velocity, and pressure, respectively. The expanded form of the convective time derivative in Eq. (1)

$$d\mathbf{V}_i/dt = (\partial \mathbf{V}_i/\partial t) + \tfrac{1}{2}\,\mathrm{grad}\, V_i^2 - \mathbf{V}_i \times \mathrm{curl}\, \mathbf{V}_i \tag{2}$$

will be used later. The last term of Eq. (1) is the friction force by electrons on the ion motion. The magnetic field strength **B** obeys the Maxwell equations

$$\mathrm{div}\ \mathbf{B} = 0, \tag{3}$$

$$\mathrm{curl}\, \mathbf{E} = -\, \partial \mathbf{B}/\partial t, \tag{4}$$

$$\mathrm{curl}\, \mathbf{B} = \mu_0(\mathbf{j} + \mathrm{curl}\, \mathbf{M}), \tag{5}$$

where μ_0 is the permeability of free space. In Eq. (5) we distinguish between the charge transport by free particles, denoted by **j**, and that by the magnetically bound or "static" particle motion, curl **M**, where the magnetization **M** is given by the density of the magnetic moments μ_e of gyrating electrons:

$$\mathbf{M} = -\, p_{e\perp}\mathbf{B}/B^2, \qquad p_{e\perp} = \tfrac{1}{2} n_e m_e V_{e\perp}^2 = n_e \mu_e B. \tag{6}$$

It is important to note the partition of free and bound electron motion, curl **H** and curl **M**, respectively, where **H** is the magnetizing field. Such a partition, which is fundamental in the classical theory of magnetized media, is often claimed to be generally unsuitable for plasmas because of the nonlinear relation between **H** and **M**. We disagree. A similar partition for the ionic motion, however, is indeed meaningless and not made here, because the evaluation of the free and definitely nonnegligible ion inertia drift current will only bring back a single-particle version of Eq. (1).

The identity

$$\frac{d}{dt}\int_{\mathbf{V}} \mathbf{a}\cdot d\mathbf{S} = \int_{\mathbf{V}} \frac{\partial \mathbf{a}}{\partial t}\cdot d\mathbf{S} + \int_{\mathbf{V}} \mathrm{div}\ \mathbf{a}\cdot \mathbf{V}\cdot d\mathbf{S} - \oint_{\mathbf{V}} \mathbf{V}\times\mathbf{a}\cdot d\mathbf{s} \tag{7}$$

together with the equations (2), (3), and (4) makes it possible to express the ion motion equation (1) in the following integral form:

$$m_i \frac{d}{dt} \oint_{\mathbf{V}_i} \mathbf{V}_i \cdot d\mathbf{s} = -e \frac{d}{dt} \int_{\mathbf{V}_i} \mathbf{B} \cdot d\mathbf{S}$$

$$- \oint_{\mathbf{V}_i} \frac{1}{n_i} (\text{grad}\, p_i) \cdot d\mathbf{s} - e \oint_{\mathbf{V}_i} \frac{1}{\sigma} \mathbf{j} \cdot d\mathbf{s}. \quad (8)$$

These integrals refer to a closed loop, length element $d\mathbf{s}$, which defines the boundary of a surface, surface element $d\mathbf{S}$. All parts of the loop follow the local ionic velocity $\mathbf{V}_i$, i.e., the loop is "frozen" into the ion gas. Note that there is no loss of generality in rewriting Eq. (1) as Eq. (8). With the vector field $\mathbf{a}$ in Eq. (7) as

$$\mathbf{a} = \mathbf{B} + (m_i/e) \,\text{curl}\, \mathbf{V}_i \quad (9)$$

and standard vector relations, the derivation of Eq. (8) is not too difficult.

In the following we shall consider the variations in time and space of a magnetic field of uniform direction, $\mathbf{B} = B\hat{\mathbf{z}}$, and azimuthal symmetry, $B = B(r)$. With an ion density distribution with the same rotational symmetry as the magnetic field, Eq. (8) can be simplified as

$$m_i \frac{\mathrm{d}}{dt} (V_{i\varphi} r) = -e \frac{d}{dt} \int_0^r B(\xi)\, \xi \, d\xi - \frac{e j_\varphi r}{\sigma}, \qquad r = r(t). \quad (10)$$

THE ELECTRON MOTION EQUATIONS

For the assumed cylindrical geometry the Maxwell equation (5) becomes

$$-\frac{1}{\mu_0} \frac{\partial B}{\partial r} = j_\varphi + \frac{\partial}{\partial r} \left(\frac{p_{e\perp}}{B} \right), \quad (11)$$

where a free-particle current j_φ is the sum of the total ion motion and azimuthal drift part of the electronic motion. In the first-order orbit theory, only the electric-field drift and the magnetic-field gradient drift enter for these electron drift motions:

$$j_{e\varphi} = e n_e \left(\frac{E_r}{B} \right) + \left(\frac{p_{e\perp}}{B^2} \right) \left(\frac{\partial B}{\partial r} \right) \quad (12)$$

and

$$j_\varphi = en_i V_{i\varphi} + j_{e\varphi}. \tag{13}$$

Equations (11), (12), and (13) give the total current

$$-\left(\frac{1}{\mu_0}\right)\left(\frac{\partial B}{\partial r}\right) = en_i V_{i\varphi} + en_e\left(\frac{E_r}{B}\right) + \frac{1}{B}\left(\frac{\partial p_{e\perp}}{\partial r}\right). \tag{14}$$

Note that the field gradient part $-(p_{e\perp}/B^2)(\partial B/\partial r)$ of the *static* magnetization current in Eq. (11) is identically cancelled by the *free* magnetic gradient drift current in Eq. (12). This paradoxical cancellation was graphically proved by Tonks[21] to be caused by the change in Larmor orbit radius with the magnetic field. It does not enter in the usual description of electrons as a massless fluid, i.e.,

$$\mathbf{E} + \mathbf{V}_e \times \mathbf{B} + (1/en_e)\,\mathrm{grad}\,p_e = 0, \tag{15}$$

which directly yields the electronic part of Eq. (14). A remarkably similar cancellation will occur among the two remaining electron terms in Eq. (14). Consider the bracket below in the expansion of these terms:

$$\frac{n_e}{B}\left[eE_r + \frac{\partial}{\partial r}\left(\frac{1}{2}\,m_e V_{e\perp}^2\right)\right] + \mu_e \frac{\partial n_e}{\partial r}\,. \tag{16}$$

In the absence of symmetry-breaking instabilities and any externally applied radial electric field, the field E_r can arise solely from the charge separation caused by differences in the radial drifts of electrons and ions so that the energy available to sustain E_r is restricted to be of the order of the electron thermal energy. To avoid an external E_r is in practice very difficult since that would require a large number of azimuthally equally distributed feeding gaps to the plasma compression coil.[22] Whatever the origin of E_r, the electron energy gained for a radial displacement, $-eE_r\delta r$, must equal the corresponding change in kinetic energy:

$$-eE_r\delta r = (\partial/\partial r)\left(\frac{1}{2}m_e V_{e\perp}{}^2\right)\delta r \tag{17}$$

and the bracket terms in (16) cancel. A further simplification, not to be made here, would be to neglect electron pressure effects altogether and thus the last term, diamagnetic current, in (16). Actually, there is such a model, called the Vlasov-Fluid Model,[23] in which the ions obey the collisionless form of Eq. (1) but the electrons are postulated to be just a massless and charge-neutralizing fluid for which

$$\mathbf{E} + \mathbf{V}_e \times \mathbf{B} = 0. \tag{18}$$

This Vlasov-Fluid Model has been proved[23] remarkably successful, better than ideal magnetohydrodynamics, in describing stability properties of high-β plasmas.

THE MAGNETIC FLUX EQUATION

The azimuthal mass velocity $V_{i\varphi}$ is eliminated between Eqs. (10) and (14). As the bracket terms in (16) cancel as discussed, the result is

$$\frac{d}{dt}\left[\lambda^2 r \frac{\partial B}{\partial r} + \mu_0\mu_e\lambda^2 r\left(\frac{\partial n_e}{\partial r}\right) - \int_0^r B(\xi)\xi\, d\xi\right] = \frac{j_\varphi r}{\sigma}, \tag{19}$$

where $\lambda = \lambda(r)$ is the collisionless ion skin depth depending on the ion density $n_i(r)$ and the ion mass m_i:

$$\lambda^{-2} = \mu_0 e^2 n_i / m_i. \tag{20}$$

For a deuterium ion density range 10^{13} - 10^{15} cm^{-3} or singly ionized argon between 2×10^{14} and 2×10^{16} cm^{-3}, for example, λ will range between 10 and 1 cm. The right-hand side of Eq. (19) is the resistive voltage drop per radian of arc of the considered loop. We make the usual assumption for magnetically confined high-temperature plasmas that such resistive electric fields are negligible compared to the inductive fields. Equation (19) can than be integrated directly:

$$\lambda^2 r \frac{\partial B}{\partial r} + \mu_0\mu_e \lambda^2 \frac{\partial n_e}{\partial r} - \Phi = C_0, \tag{21}$$

where

$$\Phi(r) = \int_0^r B(\xi)\xi\, d\xi \tag{22}$$

and, as seen from Eq. (10), the integration constant C_0 is a constant of motion and is proportional to the canonical, i.e., matter-plus-field, angular momentum of the loop at any reference instant of time t_0 when also $r = r_0$, $n_i(r_0) = n_0$, and $\lambda = \lambda_0$. For compactness, the usual factor 2π in expressions for flux, Eq. (22), and angular momentum has been dropped.

Equation (21) can be written as a linear differential equation relating Φ and r:

$$\frac{\partial^2\Phi}{\partial r^2} - \frac{1}{r}\left(\frac{\partial \Phi}{\partial r}\right) - \lambda^{-2}\left(\Phi - C_0\right) = -\mu_0\mu_e r\left(\frac{\partial n_e}{\partial r}\right) \tag{23}$$

provided that $\lambda(n_i)$ and the right-hand-side term can be expressed as known functions of r. However, $n_i(r)$ is known only to the extent of total ion number conservation within r when there is balance between ionization and recombination:

$$\int_0^r n_i(\xi)\,\xi\,\mathrm{d}\xi = \text{const in time}, \tag{24}$$

so that it will be assumed

$$n_i r^\alpha = n_0 r_0^\alpha, \qquad 1 < \alpha < \infty. \tag{25}$$

The case $\alpha = 1$ would mean a line distribution of ions, and $\alpha = 2$ is the important special case of retained relative ion density distribution everywhere upon plasma column compression or expansion, i.e., something like a "breathing" mode. The case $\alpha > 2$ means a "piling up" of ions in the vicinity of r upon compression, and $\alpha = \infty$ means perfect attachment of the ions to a cylindrical piston surface at r. The last term of Eq. (23) is readily made a known function of r by means of the quasineutrality condition $n_e = n_i$ followed by Eq. (25) and finally the constancy of motion for electronic magnetic moments. The resulting differential equation becomes

$$\frac{\partial^2\Phi}{\partial r^2} - \frac{1}{r}\left(\frac{\partial \Phi}{\partial r}\right) - \lambda_0^{-2}\left(\frac{r_0}{r}\right)^\alpha\left(\Phi - C_0 + \frac{m_i\alpha\mu_e}{e^2}\right) = 0. \tag{26}$$

For $\alpha \neq 2$ it has the solution

$$\Phi(r) = C_0 - m_i\alpha\mu_e/e^2 + rA_1I_\nu(s) + rA_2K_\nu(s),$$

$$\nu = 2/(\alpha-2), \qquad s = s(r) = (\nu r_0/\lambda_0)\,(r/r_0)^{1/\nu}, \tag{27}$$

where I_ν and K_ν are the ν-th-order modified, also called hyperbolic, Bessel functions of the first (I) and second (K) kind. To determine the space integration constants A_1 and A_2 with the use of the time integration constant C_0, the latter has to be split up into its flux part Φ_0 and its mechanical angular momentum part L_0,

$$C_0 = \Phi_0 + L_0, \quad \Phi_0 = \int_0^{r_0} B(\xi,\, t_0)\,\xi\, d\xi,$$
$$L_0 = (m_i/e)\,V_{i\varphi 0}r_0, \tag{28}$$

and, further, $B(r)$ has to be calculate£d from Eq. (27). With the use of modified Bessel function recurrence formulas it is found that

$$B(r) = \frac{1}{r}\left(\frac{\partial\Phi}{\partial r}\right)$$
$$= \frac{1}{\lambda_0}\left(\frac{r}{r_0}\right)^{1/\nu-1}\ [A_1I_{\nu-1}(s) - A_2K_{\nu-1}(s)]. \tag{29}$$

A_1 and A_2 can now be determined from Eqs. (27) and (29) as

$$A_1 = \nu[B_0r_0K_\nu(s_0) - \lambda_0^{-1}(L_0 - m_i\alpha\mu_e/e^2)K_{\nu-1}(s_0)],$$
$$A_2 = -\nu[B_0r_0I_\nu(s_0) + \lambda_0^{-1}(L_0 - m_i\alpha\mu_e/e^2)I_{\nu-1}(s_0)]. \tag{30}$$

From Eqs. (11), (29), and recurrence relations the total current density is obtained as

$$-\frac{1}{\mu_0}\left(\frac{\partial B}{\partial r}\right) = \frac{1}{\mu_0\lambda_0^2}\left(\frac{r}{r_0}\right)^{2/\nu-2}[A_1I_\nu(s) + A_2K_\nu(s)]. \tag{31}$$

With the solution Eq. (26) and canonical angular momentum conservation at any r, i.e.,

$$\Phi(r) + (m_i/e)\,V_{i\varphi}(r)\,r = C_0, \tag{32}$$

we, as expected, reproduce from Eq. (31) the discussed partition into free and bound current densities:

$$-\frac{1}{\mu_0}\left(\frac{\partial B}{\partial r}\right)=\frac{1}{\mu_0\lambda_0^2 r_0}\left(\frac{r}{r_0}\right)^{-\alpha-1}\left(\frac{m_i}{e}V_{i\varphi}r-\frac{\alpha m_i}{e^2}\mu_e\right)$$
$$=en_iV_{i\varphi}+\mu_e\frac{\partial n_e}{\partial r}. \tag{33}$$

For $\alpha = 2$, i.e., uniform compression or expansion, the solution of Eq. (26) is a simpler algebraic expression:

$$\Phi(r) = C_0 - 2m_i\mu_e/e^2 + r(A_1r^q + A_2r^{-q}), \tag{34}$$

where

$$q^2 = 1 + r_0^2/\lambda_0^2,$$

$$A_1 = (1/2q)[B_0r_0^{-q+1} - (L_0 - 2m_i\mu_e/e^2)(q-1)r_0^{-1-q}],$$

$$A_2 = -(1/2q)[B_0r_0^{q+1} + (L_0 - 2m_i\mu_e/e^2)(q+1)r_0^{q-1}]. \tag{35}$$

The effect of the "static" electronic motion owing to the electron density gradient appears as a correction ($-m_i\alpha\mu_e/e^2$) to the mechanical angular momentum. Under typical theta-pinch conditions, e.g., an electron temperature of 100 eV, $B = 1$ Wb/m^2, $r = 10$ cm and deuterium ions, this term is about three powers of ten smaller than the flux part, $\approx Br^2/2$, of the canonical angular momentum. However, it may be large in a region of vanishing field strength where, of course, our first-order orbit description of the electronic motion is no longer valid. In a Vlasov-Fluid Model of the plasma[23] this electronic correction vanishes, but in the opposite limit, where the plasma is taken as a single-species quasineutral fluid,[16] it is implicitly taken to cancel its ionic counterpart.

LIMITING CASES

It is to be emphasized that the fields and fluxes obtained so far express no more than conservation of canonical angular momentum for that certain circular loop in the moving ion gas frame that is characterized by the set of initial conditions r_0, B_0, Φ_0, etc. Singularities of I_ν and K_ν at $s \to \infty$ and $s = 0$, respectively, simply mean that infinite expansion or compression of this loop would require infinite field strengths and fluxes.

As assumed, the plasma channel tends with increasing α, i.e., $\nu \to \infty$ in Eqs. (27), (29), and (30), toward the physically nonexistent "sharp boundary" model popular in stability theories. Specifically, for that large value,

$$\alpha = eV_{i\varphi 0}r_0/\mu_e, \tag{36}$$

a particular solution directly found from the differential equation (26) is $\Phi = \Phi_0$. It may be pointed out that this would not be a case of the Lighthill theorem[24] of flux conservation in the electron gas frame; actually, it is the "static" diamagnetic current taken as negligible in his derivation that would preserve the initial flux Φ_0 within r.

The classical MHD theorem of flux conservation in the mass frame, i.e., "frozen" field lines, is found already in Eq. (19) for $\sigma \to \infty$ and $\lambda = 0$. Note the ambiguity of the latter limit; it is obtained both for $m_i \to 0$ and for $n_e = n_i \to \infty$. This is consistent with the usual explanation of the theorem as a consequence of

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \mathbf{j}/\sigma, \qquad \sigma \to \infty, \tag{37}$$

where $\mathbf{V}$ has to be both the mass velocity and some net charge velocity.

The striking feature of Eq. (34) is its very strong dependence on the initial ion density contained in the parameter $r_0/\lambda_0(n_0)$. There is a similar strong, but less obvious, dependence in the Bessel function expressions. The striking feature in experimental work on theta-pinch dynamics (see, e.g., Ref. 12) is indeed also the very strong influence of the initial ion density.

In the limit $\lambda_0 \gg r_0$ it is found with the aid of the following limiting forms, involving the Γ-function $\Gamma(\nu + 1) = \nu\Gamma(\nu)$, for small arguments:

$$I_\nu(s) \approx \frac{[(1/2)s]^\nu}{\Gamma(\nu+1)}, \qquad \nu \neq -1, -2, \ldots \tag{38}$$

$$K_{-\nu}(s) = K_\nu(s) \approx \tfrac{1}{2}\Gamma(\nu)(\tfrac{1}{2}s)^{-\nu}, \qquad \nu \neq 0, \tag{39}$$

that

$$\Phi(r) = \frac{r^2}{2}\left[B_0 - \frac{(L_0 - m_i\mu_e/e^2)\nu}{\lambda^2_0\, 2(\nu-1)}\right] \approx \frac{r^2 B_0}{2}. \tag{40}$$

In the $\alpha = 2$ case, Eqs.(34) and (35) yield consistently with Eq. (40):

$$\Phi(r) = \frac{r^2}{2}\left[B_0 - \frac{1}{2\lambda_0^{\,2}}(L_0 - 2m_i\mu_e/e^2)\right] \approx \frac{r^2 B_0}{2}\,. \tag{41}$$

The singularity $\nu = 1$ in Eq. (40) is probably nonphysical since it can be traced back to the nonvalidity point stated in Eq. (38). Physically, the results Eqs. (40) and (41) are almost trivial. When $\lambda_0 \gg r_0$ the ions are so few and heavy that the inductive effects associated with their motion from r_0 to r are insufficient to change the initial field B_0.

In the opposite limit, $\lambda_0 \gg r_0$, i.e., s and $s_0 \gg 1$ when ν is finite, the limiting forms of I_ν and K_ν for large arguments,

$$I_\nu(s) \approx \exp s/(2\pi s)^{1/2}, \qquad K_\nu(s) \approx \frac{\exp(-s)}{(\pi/2s)^{1/2}}\,, \tag{42}$$

give for the flux of Eq. (27)

$$\begin{aligned}\Phi(r) = C_0 - &\frac{m_i\alpha\mu_e}{e^2}\\ &+ \left(\frac{r_0}{r}\right)^{1/2\nu - 1}\Big[B_0 r_0\lambda_0 \sinh(s - s_0)\\ &- \left(\frac{L_0 - m_i\alpha\mu_e}{e^2}\right)\cosh(s - s_0)\Big].\end{aligned} \tag{43}$$

A linearization with respect to $s \approx s_0$ proves that a radial shift δr is accompanied by a flux change $\delta\Phi$ similar to that obtained in Eqs. (40) or (41):

$$\delta\Phi = B_0 r_0 \delta r, \qquad \delta\Phi = \Phi(r_0 + \delta r) - \Phi(r_0). \tag{44}$$

This result, however, is almost meaningless since it is a consequence of the double and too extreme inequality $\delta r \ll \lambda_0 \ll r_0$. Equation (43) can have physical relevance only when the steep and large hyperbolic functions essentially balance each other, as occurs already for $\delta r \approx \lambda_0$:

$$|B_0 r_0 \lambda_0| \approx |L_0 - m_i\alpha\mu_e/e^2|. \tag{45}$$

The factor $\lambda_0 \ll r_0$ in Eq. (45) means that the mechanical part of the total canonical angular momentum is small when n_i is large because of large flux-creating ionic current then carried at a small or moderate ionic velocity. Equation (45) also shows that the initial values B_0, L_0, etc., are not in general independent as they have to comply with some reasonable physical process creating such an initial set.

"MAGNETIC PISTONS" AND OTHER PHENOMENA

It is a well-known but intriguing feature of theta pinches that the moving region of steep magnetic field strength variation, commonly interpreted and referred to as a "magnetic piston," may precede or lag behind the moving region of steepest increase or maximum of plasma density.[6, 7, 12] The sign of this discrepancy depends on the initial plasma density, and the change occurs for an initial deuterium fill density in the region 10^{14}–10^{15} cm^{-3}. Consider Eq. (34). For inward plasma motion the associated field strength varies essentially as $(r_0/r)^q$; however, the plasma density has been assumed to vary as $r^{-\alpha} = r^{-2}$. A probe operator would record a "magnetic piston" *preceding* the imploding plasma front for

$$q = (1 + r_0^2/\lambda_0^2)^{1/2} \geqslant 2, \qquad \text{i.e., } r_0^2 \geqslant 3\lambda_0^2. \tag{46}$$

For $r_0 = 6$ cm, for example, deuterium ions and absence of impurities as well as ion coupling to neutral atoms, the inequality (46) leads to $n_{i0} \geqslant 10^{13}$ cm^{-3} in reasonable agreement also quantitatively with observations.[12] In the Bessel function expression (29) the function $K_{\nu-1}(s)$ accounts for the corresponding "magnetic piston" when $\alpha > 2$. Roughly, it becomes significant when $s_0 \geqslant 1$:

$$s_0 = \nu r_0/\lambda_0 \geqslant 1, \qquad \text{i.e., } r_0/\lambda_0 \geqslant \alpha/2 - 1, \tag{47}$$

which is essentially the same result as Eq. (46). For $\alpha < 2$, so that $\nu < 0$, the roles of $I_{\nu-1}(s)$ and $K_{\nu-1}(s)$ are reversed. but, again, the parameter λ_0 has to be of the order r_0 or less:

$$-\nu r_{0/\lambda_0} \geq 1, \qquad \text{i.e., } r_0/\lambda_0 \geq 1 - \alpha/2. \tag{48}$$

"Magnetic pistons" of various elaborate types (sharp, diffuse, leaking, reflecting, snowplowing, hybrids, etc.) appear more and more in the literature. Harold Grad[13] has explained why the Bohm diffusion concept actually is "a degradation of information." We think this view and most of his arguments for it also apply to the "magnetic piston" concept.

One of the most extensive and careful experimental investigations of theta-pinch implosion dynamics with special regard to the rotation origin has been given by James Benford in Ref. 7. As any electromagnetic origin was claimed to be impossible already in the introduction, the outcome of his investigation is, as expected, a veritable massacre of all proposed mechanisms except for one theory[25] based on computer calculations that seemed to agree qualitatively but failed quantitatively. More emphasis, however, was given in his paper to the observation of a strongly varying radial velocity distribution, which (we note) strongly conflicts with the often assumed and reported rigid-body rotation.[17,18] During the implosion phase the ion velocity was found to change direction near the region of the plasma density maximum. Further, although resistive effects were far from negligible, contrary to that assumed here, an appreciable part of the current was found to be ionic. It seems that the parameter r_0/λ_0 in Ref. 7 was probably a few times unity ($n_i = 10^{16}$ cm^{-3} for nitrogen gives $\lambda_0 = 1$ cm) so that the observed separation of the field gradient from and ahead of the density gradient is as expected. The conditions for ion velocity reversal, easily found to be essentially $A_1A_2 < 0$, $|A_2| < |A_1|$, for Eq. (34) in combination with Eq. (32), may have been satisfied, but the data presented do not allow the estimation of an initial set B_0, L_0, etc., with any reliability.

The transfer of mechanical angular momentum into magnetic flux is a dynamo mechanism. It can also be considered as a Barnett effect for plasmas. It has been experimentally observed for a rotating plasmoid ejected out of a plasma gun,[26] and specific features in the spontaneous generation of magnetic fields in laser-produced plasmas indicate that the phenomenon exists there.[27]

STATIC AND MOVING-FRAME ENERGIES

Consider a coaxial cylinder section of the plasma column. It is taken to have an arbitrary but nonmoving radius R and to be of unit length in the axial z direction. The total power influx through the cylindrical surface is obtained from the Poynting vector $\mathbf{E} \times \mathbf{H}$ as

$$P_{\text{in}} = (-\hat{\mathbf{r}}2\pi R) \cdot (\hat{\boldsymbol{\varphi}}E_\varphi(R)) \times (\hat{\mathbf{z}}H_z(R)) = -2\pi R E_\varphi H_z. \tag{49}$$

The azimuthal symmetry makes E_φ purely inductive:

$$E_\varphi = -\partial A_\varphi / \partial t, \tag{50}$$

where **A** is the magnetic vector potential,

$$\mathbf{B} = \operatorname{curl} \mathbf{A}, \tag{51}$$

and **A** is related to the flux Φ as

$$A_\varphi(r) = \Phi(r)/r, \qquad \Phi(r) = \int_0^r B(\xi)\,\xi\, d\xi, \qquad A_r = A_z = 0. \tag{52}$$

During a time interval t_0–t the total energy $W_{\text{in}}(R)$ fed through the plama cylinder surface $r = R$ is

$$W_{\text{in}} = -2\pi R \int_{t_0}^{t} H_z(R)\,E_\varphi(R)\,dt$$

$$= 2\pi \int_{t_0}^{t} \Big[\frac{1}{\mu_0 R} \Big(\frac{\partial \Phi}{\partial R} \Big) - M \Big] \Big(\frac{\partial \Phi}{\partial t} \Big) dt. \tag{53}$$

Now, a quantity W_{me} is defined and expanded as

$$W_{\text{me}}(R) = -\int_{t_0}^{t}\int_0^R 2\pi r \mathbf{j} \cdot \Big(\frac{\partial \mathbf{A}}{\partial t} \Big) dt\, dr$$

$$= \int_{t_0}^{t}\int_0^R 2\pi\ \frac{\partial}{\partial r} \Big[\frac{1}{\mu_0 r} \Big(\frac{\partial \Phi}{\partial r} \Big) - M(r) \Big] \Big(\frac{\partial \Phi}{\partial t} \Big) dt\, dr. \tag{54}$$

A partial space integration yields

$$W_{\text{me}}(R) = W_{\text{in}}(R) - \int_{t_0}^{t}\int_0^R 2\pi \Big[\frac{1}{\mu_0 r} \Big(\frac{\partial \Phi}{\partial r} \Big) - M(r) \Big] \frac{\partial}{\partial r} \Big(\frac{\partial \Phi}{\partial t} \Big) dt\, dr, \tag{55}$$

and this integral is readily recognized as that part of $W_{\text{in}}(R)$ that has been stored within R as magnetic energy:

$$W_{\text{ma}}(R) = \int_{t_0}^{t}\int_0^R 2\pi r \Big[\frac{1}{\mu_0 r} \Big(\frac{\partial \Phi}{\partial r} \Big) - M(r) \Big] \frac{\partial}{\partial t} \Big[\Big(\frac{1}{r} \Big) \Big(\frac{\partial \Phi}{\partial r} \Big) \Big] dt\, dr$$

$$= \int_{t_0}^{t}\int_0^R 2\pi r\, H_z(r)\left(\frac{\partial B_z}{\partial t}\right) dr\, dt. \tag{56}$$

The MHD approximations made, in particular neglect of displacement currents and quasineutrality, make radiation and electric field energies negligible compared to W_{ma}. Also, in the usual Amperian formulation of electrodynamics that we use, W_{ma} includes the non-negligible energy transfer to the magnetically bound particles (see Ref. 19, p. 211). Thus W_{me} means the total mechanical energy imparted to the *ions* in the volume $r < R$ during $t_0 - t$. A mechanical energy density $w_{\text{me}}(R)$ can therefore be associated with $W_{\text{me}}(R)$ as

$$w_{\text{me}} = \left(\frac{1}{2\pi R}\right)\frac{\partial W_{\text{me}}}{\partial R} = -\int_{t_0}^{t} \mathbf{j}(R)\cdot\frac{\partial \mathbf{A}(R)}{\partial t}\, dt. \tag{57}$$

In the moving ion gas frame, velocity $V_{ir}\hat{\mathbf{r}} + V_{i\varphi}\hat{\boldsymbol{\varphi}}$, however, $-\mathbf{j}\cdot(d\mathbf{A}/dt)$ replaces $-\mathbf{j}\cdot(\partial\mathbf{A}/\partial t)$, i.e., the mechanical energy transfer from the $\mathbf{V}_i\times\mathbf{B}$ field must be included. In the absence of charge accumulation, i.e., $j_r = 0$, this is seen by expanding $d\mathbf{A}/dt$:

$$\begin{aligned} w'_{\text{me}}(r, t) &= -\int_{t_0}^{t} \mathbf{j}\cdot\left(\frac{d\mathbf{A}}{dt}\right) dt \\ &= -\int_{t_0}^{t} \mathbf{j}\cdot\left[\frac{\partial\mathbf{A}}{\partial t} + (\mathbf{V}_i\cdot\nabla)\mathbf{A}\right] dt \\ &= -\int_{t_0}^{t} \mathbf{j}\cdot\left(\frac{\partial\mathbf{A}}{\partial t}\right) dt + \int_{t_0}^{t} \mathbf{j}\cdot(\mathbf{V}_i\times\mathbf{B})\, dt. \end{aligned} \tag{58}$$

BIAS FIELDS

The early implosion phase in magnetic compression is considered and, in particular, the inflow to the plasma of compression work plus radially directed kinetic energy. The derived mechanical energy density w'_{me} is therefore reduced by the energy density stored in azimuthal rotation:

$$w_{me}^{*}(r) = -\int_{r_0}^{r} j(\xi)\, dA(\xi) - \tfrac{1}{2} m_i n_i(r) V_{i\varphi}^{2}(r)$$

$$= -\frac{e^2}{m_i}\left[\int_{r_0}^{r} n_i \frac{C_0 - \Phi}{\xi}\left(\frac{\partial}{\partial \xi}\right)\left(\frac{\Phi}{\xi}\right) d\xi + \tfrac{1}{2} n_i \frac{(C_0 - \Phi)^2}{r^2}\right]. \quad (59)$$

A positive energy-transfer rate during compression to the moving plasma unit volume is given by the product of two negative quantities, V_{ir} and $\partial w_{\mathrm{me}}^{*}/\partial r$,

$$V_{ir} \frac{\partial w_{\mathrm{me}}^{*}}{\partial r} = V_{ir} \frac{e^2 n_i \alpha}{2 m_i r^3} (\Phi - C_0)(\Phi - C_0 - \frac{2}{\alpha} C_0). \quad (60)$$

It is assumed that plasma is initially at rest and permeated by the axial and uniform bias field B_0,

$$C_0 = \Phi(r_0) = \frac{1}{2} B_0 r_0^{2},$$

$$\Phi(r) - C_0 \geqslant 0 \qquad \text{for } r \leqslant r_0. \quad (61)$$

Equation (60) predicts a remarkable phenomenon: Compression in combination with a positive bias field would require that w_{me}^{*} decreaseñ Clearly, this is impossible because the plasma was taken to have no such energy initially. The only reasonable solution is $V_{ir} = 0$, i.e., the positively biased plasma column simply does not compressñ Equation (60) does not deny compression altogether—it will occur *but not until the main driving field has impressed an additional flux of the same order as the bias flux or $2C_0/\alpha \approx \Phi_0$*. For zero or a negative bias field Eq. (60) states that compression and ion heating start immediately as the main driving field is applied.

We have previously shown in Ref. 28 how an important partition between plasma magnetic, kinetic, and thermal energies could be accomplished by considering the plasma as an assembly of a large number of magnetically and mechanically interacting but separate current loops. Note that the old and elaborate electrodynamics of such quasistationary linear current systems[29] allows the loops to move in an arbitrary (but nonferromagnetic) dielectric. A both central and remarkable result from this theory is that the evolved magnetic energy acts as a *negative* potential energy for mechanical energy. Omitting the proof, we just state from such a loop reasoning our physical interpretation of the above obtained, and experimentally well-known, result of bias-field influence. To establish a positive bias field is a waste of potential energy that could have been used for heating and compression. The negative bias field acts instead as a potential energy reservoir.

BIAS MECHANICAL ANGULAR MOMENTUM

According to Eq. (60) the described results about bias-field influence apply to any initial canonical angular momentum regardless of its composition into flux and field parts. We may then assume that a cylindrical discharge vessel is rotating like an isotope separation centrifuge of 0.1 m radius and a peripheral velocity of 500 m/s. A deuterium atom that follows the wall rotation and becomes ionized carries a mechanical angular momentum that, translated according to Eq. (28) into a *uniform* magnetic flux across the centrifuge cross section, corresponds to a field strength of 2×10^{-4} Wb/m^2 = 2 G. This negligible field strength, however, cannot be used to refute fusion-technology use of bias angular momentum. Strictly, the result applies only to the case $\lambda_0 \gg r_0$ discussed above under the heading "Limiting Cases." For $\lambda_0 \leqslant r_0$ only an annular strip of the width λ_0 is available for magnetic flux generation [cf. Eq. (45)]. Then, the same initial rotation would correspond to the small but nonnegligible bias-field strength of 100 G if there were an initial ion density of 10^{17} cm^{-3} near the wall, i.e., $\lambda_0 = 1$ mm.

A physical explanation of the effect of a negative bias angular momentum is almost trivial: The moving-frame electric field of the external driving magnetic field is then directed both so as to *increase* the initial velocity of the ions and so as to deflect them radially inwards.

Tentatively, we indicate a few reasons for substituting the bias magnetic field by bias angular momentum or a combination of both:

1. More pronounced thermonuclear conditions may be expected from an imploding plasma that retains an initial shell structure. Such a plasma of finite length and sufficient density can be made

to create a field reversal ahead of the imploding plasma front. The resulting configuration would then be an "Ion Ring Compressor"[30] but charge-neutralized and without a separate ion injection arrangement. (The "Ion Ring Compressor" is topologically the old Astron concept but with an ion ring replacing the electron beam layer.)

2. At least in principle, bias angular momentum is more flexible than the topologically rather constrained irrotational bias field. ($\mathbf{B}_0$ always varies in *both* strength and direction.)
3. The strongly stabilizing action of externally impressed rotation upon straight discharges may exist also for theta pinches.

SUMMARY AND CONCLUSIONS

The standard equation (1) for ion mass motion was rewritten, without approximations, so as to express by Eq. (8) the experimentally observed mass rotation for the ion gas as the externally applied field permeates the theta-pinch plasma. It has been pointed out previously[20] that this rotation origin, in contrast to the other proposed mechanisms, accounts for the strong and early rotation onset. For consistency the electronic motion was described both in fluid formulation and by first-order orbit expressions valid for the assumed conditions of strong fields and negligible collisions. Hence, phenomena associated with zero field regions[11] in the plasma, or conditions at the end regions[8] where the assumed simplified azimuthal symmetry is not satisfied, are not included in our analytical model of the plasma. The two-particle species equations were combined into Eq. (19), which relates plasma magnetic flux and plasma radial motion. The solutions of this equation were discussed and found to depend strongly upon a length ratio λ_0/r_0, which can also be expressed as the ratio of ambient Alfvén wave velocity and ionic Larmor gyration speed.

The aim and the essence of the analysis have been to show that basic plasma equations combined with well-known results and techniques from classical electrodynamics of moving media[19] do predict at least qualitatively a number of experimentally observed phenomena like plasma rotation, rapid field diffusion, separation field plasma, rotation reversal, and peculiar bias-field influence. For a quantitative description, analytical techniques like the present one do not seem feasible. In particular, important microprocesses like ionization, recombination, and excitation of experimentally observed instabilities can probably be handled only by extensive numerical techniques.

NOTE ADDED IN PROOF

It has been argued that the "well-established" or "universally accepted" fast-theta-pinch description refutes our very different description. The essence of this "experimentally proved" description seems to be the following: When subject to an externally impressed and very rapidly varying magnetic field the heavy ions at first remain stationary. They are then dragged radially inwards by a strong electrostatic field E_r that is created by radially, $V_{er} \approx E_\varphi/B$, and azimuthally, $V_{e\varphi} \approx E_r/B$, *drifting* electrons. We note about such a description:

i. An *azimuthally* directed ion velocity will always be imparted to the ions however short the rise time is of the impressed field. See Ref. A-1 for a calculation of the remarkable single-ion trajectories when the field rise time is much smaller than the ion gyration time.
ii. As *drifting* electrons by the very definition of drift velocity acquire very limited energy, they are clearly unable energetically to "drag" heavy ions up to the observed fusion kiloelectron volt conditions.
iii. Detailed and careful electrical probe measurements of E_r do not substantiate the "well-established" theory. See, e.g., Ref. A-2.

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