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Production of Fusion Energy By Vortex Structure Compression

D.R. Wells and P. Ziajka

On Integrals of the Hydrodynamic Equations That Correspond to Vortex Motions

Hermann von Helmholtz

Translated by Uwe Parpart

RESEARCH REVIEW

Some New Directions in Fluid Mechanics

Steven Bardwell

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Production of Fusion Energy by Vortex Structure Compression

D.R. WELLS AND P. ZIAJKA

INTRODUCTION

The first thought man gave to ionized gases probably occurred when one of our cave-dwelling ancestors marveled at a lightning bolt. Since then, the form and substance of ionized gases have always intrigued thinking men. As early as the 16th century, the Englishman William Gilbert made a scientific investigation into the effect of a magnetic field on a candle flame.

The discovery and development of electricity led a few early physicists to study the interaction of electromagnetic forces and gases. Sir William Crooke, in the late 19th century, developed the "glow-discharge tube" and was apparently the first to realize that the electrified gas was no longer truly a gas but was actually a new, fourth state of matter. The discovery and exploration of the ionosphere in the early part of this century did much to promote interest in the behavior of ionized gas. In 1928, Irving Langmuir, working at the General Electric labs, coined the word "plasma" to describe electrified gases in a vacuum tube.

It was not until the late 1940s, however, that the science of plasma physics really began to receive serious attention. At that time, shortly after the hydrogen bomb explosions at Aniwetok Atoll in the West Pacific, it became obvious to physicists that an essentially unlimited source of energy would be available if the thermonuclear reaction could be controlled. To this end, the U.S. government established a classified project, Project Sherwood, in 1951. After a series of technological and theoretical problems arose, the hope of an early success was dismissed, the project was declassified, and a broad spectrum of approaches to the problem of heating and confining a thermonuclear plasma was evolved by plasma physics scientists. Limited only by imagination and budget, researchers tried scores of heating and confinement configurations, with such intriguing names as stellerator, tokamak, and Alcator. The effort was international. The Soviet Union, USA, European nations, and Japan all initiated programs, and there was and still is a great deal of international cooperation and interchange of personnel among these efforts.

This article is concerned primarily with one approach: the heating and confinement of stable, force-free plasma vortex structures produced by a theta-pinch gun.

VORTEX MOTION IN PLASMAS

Several researchers observed the rotation in plasma produced by theta pinches, and they put forward various hypotheses to explain the origin of this rotation[1-6]. An intensive experimental investigation of conical theta pinches by Wells in 1962[7] revealed that the observed rotation inside the pinch coil was associated with a radial electric field (There was no rigid body rotation) and that the plasma in the throat of the theta pinch had a net positive potential. Another investigation by Wells[8-9] was concerned with plasma structures produced by a conical theta-pinch coil placed inside a magnetic guide field. Using magnetic probes, Wells determined that the plasma geometry of these structures was toroidal, with a 1,000 gauss trapped poloidal field and 200 gauss trapped toroidal field. A 25 KV theta-pinch coil was used with a 2.5 microsecond quarter cycle rise time and a 4,000 gauss solenoidal guide field. It was first speculated in the report on this work that this plasma structure was a plasma vortex ring, but the theory of this vortex formation was not developed until late 1963.

The following sections will deal with the process of the development and elaboration of this theory and its experimental confirmations.

Since 1962, various researchers and laboratories have reported ordered plasma motion that can be classified as vortex phenomena. In addition to the initial report and subsequent confirmations by Wells, et al.[7-14], vortex structures have been reported by Farber, Prior, and Bostick[15], Hogberg and Vogel[16], Waniek[17], Komilkov, et al.[18], Komen et al.[19], and Jones and Miller[20].

Before going into the details of plasma vortex structures produced in a conical theta pinch, we will present an abbreviated review of vortex formation and development in general.

THE CONCEPT OF VORTICITY IN MHD AND PLASMA PHYSICS

The concept of vorticity plays a major, well-known role in hydrodynamics and aeromechanics. The rapid and important developments in aeronautical science have been based primarily on the classical work accomplished in hydrodynamics in the 19th century. Hydrodynamics is a branch of continuum mechanics in which a fluid is treated as a macroscopic entity and it deals primarily with the laws and rules that control a fluid element in a surrounding fluid environment. It is not necessary to treat the microscopic aspects of the problem at all, except as they determine the macroscopic laws involved in the analysis.

Although irrotational flow is an interesting and important subject in itself, the subject of hydrodynamics becomes very exciting when one considers rotational flow with its vortex filaments and cores. This aspect of the theory allows the aerodynamicist to study lift and other basic phenomena in aircraft design and theory.

The plasma physics community has not neglected the concept of continuum mechanics, but utilizes the magnetohydrodynamic model, the analogue of hydrodynamics. Unfortunately, most current applications of magnetohydrodyamics completely neglect the vortex concept. The reasons are not entirely clear, although one suspects that the training and background of most physicists working in plasma physics do not include much exposure to this aspect of the problem. For some reason, MHD is a respected and widely used model, but the mere mention of vorticity or a rotational MHD model seems to evoke an almost irrational, negative response. In fact, it is considereed stylish to ridicule a rotational fluid in any discussion of plasma physics or magnetohydrodynamics.

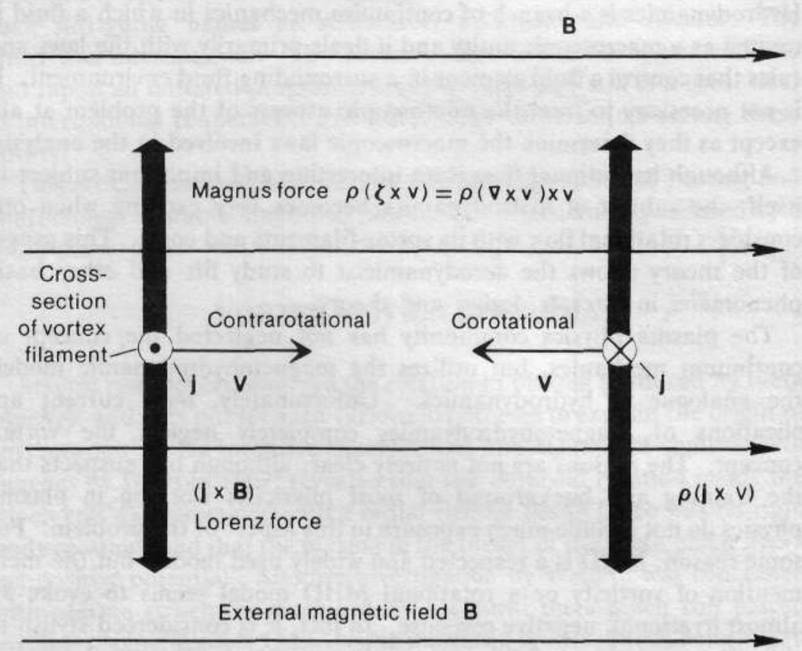
In spite of these apparent prejudices, one can easily show that very powerful tools can be developed by including the concept of vorticity in the MHD model. These prove to be just as important to MHD as the rotational normal fluid is in aerodynamics.

COROTATIONAL AND CONTRAROTATIONAL PLASMA STRUCTURES

The first important work on rotational MHD fluids was by Busemann in the late 1940s[21]. An aerodynamicist who worked with Von Karman and others in the development of supersonic flow theory, Busemann considered the laws of continuum mechanics as they applied to vortex filaments in an MHD fluid.

Consider a vortex filament in a surrounding conducting fluid permeated by a constant magnetic field. Simple observation of the equilibrium of the filament indicates that there must be two separate and distinct types of filament (see Figure 1). The first is called a corotational vortex filament and it moves antiparallel to the magnetic field, $\vec{B}$, with velocity $\vec{v}$. A conduction current, $\vec{j}$, flows along the vortex filament and $\vec{j}$ is parallel to $\vec{\zeta}$, where $\vec{\zeta}$ is the vorticity of the filament, which is given by $\vec{\zeta} = \text{curl}\,\vec{v}$. The other type, called a contrarotational vortex filament, moves parallel to the magnetic guide field, $\vec{B}$, and has its conduction current density, $\vec{j}$, antiparallel to its vorticity, $\vec{\zeta}$. Thus, there are two basic, separate and distinct types of MHD vortices, one

FIGURE 1



The two basic types of plasma vortex filaments, corotational and contrarotational.

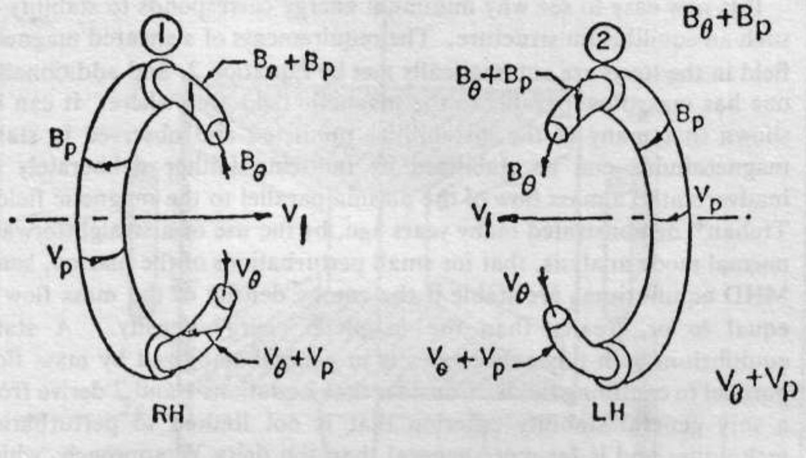
moving parallel to the background magnetic field and the other antiparallel to the background magnetic field. The corresponding balance of forces for each vortex, shown in Figure 1, accounts for the equilibrium of each simple vortex structure.

VORTEX RINGS, FORCE-FREE FIELDS, AND STABILITY

The step from the simple plasma filaments described in Figure 1 to a set of corotational and contrarotational vortex rings is simply one of going from a two-dimensional description of these structures to a three-dimensional description in spheroidal or toroidal geometry. The details of the theory of this transition are given elsewhere[10-12]. It turns out that a contrarotational ring does indeed propagate parallel to a magnetic guide field and that a corotational ring propagates in the antiparallel direction.

The equilibrium of these rings is self-evident, but the question of stability has to be considered separately. At this point in a discussion of these configurations, the work of Chandrasekhar and Woltjer plays a major role. Chandrasekhar asked what configuration the velocity and magnetic induction fields in a rotational plasma structure must take if the structure is to have minimum total energy or, in the case of a

FIGURE 2



Corotational and contrarotational force-free plasma vortex rings.

compressible fluid, free energy. The answer to this question is the following set of Euler-Lagrange equations that govern the flow and magnetic fields in such a structure.

$$\nabla \times \vec{B} = K\vec{B} \quad (1)$$

$$\vec{V} = \pm \beta \vec{B} \quad (2)$$

Equation 1 specifies a Lorentz force-free field. The $\vec{j} \times \vec{B}$ forces are everywhere zero both inside the structure and in the surrounding fluid. Equation 2 says that the mass flow field must be everywhere parallel or antiparallel to the local magnetic induction field. But Equation 2 is just the equation that describes a corotational and contrarotational plasma ring. Adopting Equation 1 to the set of rings is slightly more difficult, but the result is surprisingly simple. It is a set of force-free collinear rings, shown in Figure 2.

The core of the rings has a purely toroidal flow and corresponding current density and vorticity; that is, they are directed around the center of the ring. At the surface of the ring the flow has no toroidal component at all, but the velocity, current density, and magnetic and vortex fields all circle the cross section as poloidal fields. This, of course, is necessary to satisfy the boundary conditions, since the rings are moving parallel and antiparallel to the magnetic guide fields in the laboratory frame of reference. Between the core of the ring and surface of the ring, one then has a sheared magnetic field, velocity, current density, and vortex filament field in order to make the transition from purely toroidal at the center of the ring cross section to purely poloidal at the outside of the ring cross section.

It is now easy to see why minimum energy corresponds to stability in such an equilibrium structure. The requirements of a sheared magnetic field in the torus are automatically met by Equation 2, and additionally, one has mass flow parallel to the magnetic field everywhere. It can be shown that many of the instabilities predicted and observed in static magnetofluids can be stabilized by inducing (either deliberately or inadvertently) a mass flow of the plasma parallel to the magnetic field. Trehan[44] demonstrated many years ago, by the use of a straightforward normal mode analysis, that for small perturbations of the plasma, some MHD equilibriums are stable if the energy density of the mass flow is equal to or greater than the magnetic energy density. A static equilibrium with unstable modes is in general stabilized by mass flow parallel to confining fields. Consider that Equations 1 and 2 derive from a very general stability criterion that is not limited to perturbation techniques and is far more general than the delta W approach, which also predicts that sheared magnetic fields stabilize plasma equilibriums.

Production in the laboratory of a set of collinear force-free or quasi force-free corotational and contrarotational vortex rings should be an interesting approach to the attainment of a very hot, dense plasma.

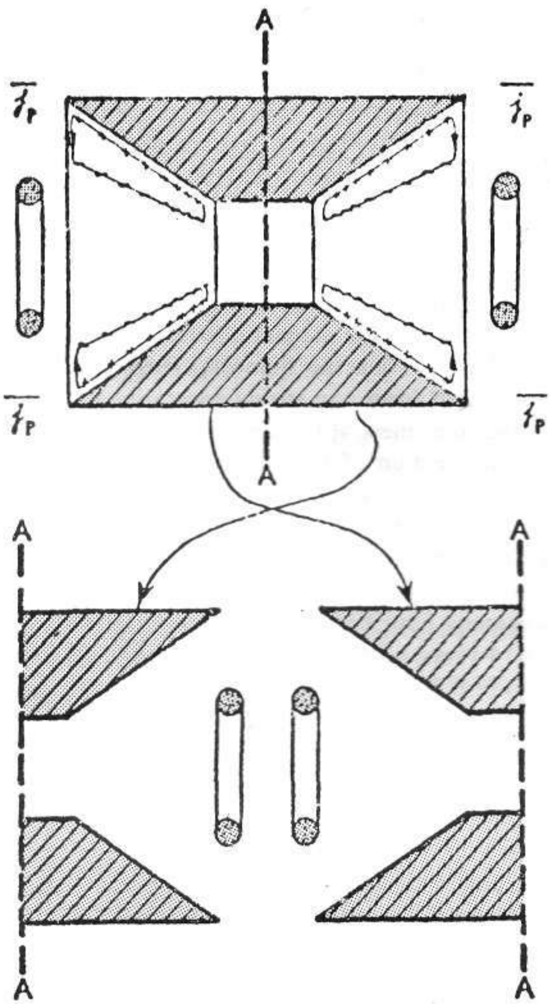
PRODUCTION OF COLLINEAR FORCE-FREE STRUCTURES IN THE LABORATORY

Consider how the structures described above might be generated in the laboratory. Wells described a method of doing this 17 years ago when he was working on a PhD dissertation at the Plasma Physics Laboratories of Princeton University. Fortunately, it turns out that a simple conical theta pinch immersed in a magnetic field parallel to the center line of the pinch coil will automatically generate the structures described in Figure 2.

Figure 3 shows a double-ended conical theta-pinch coil with a magnetic field parallel to its axis. If a current is pulsed through such a coil, a pair of plasma rings will be generated at each end of the coil. It is easily shown that the right-hand ring will be corotational and the left-hand ring will be contrarotational. The toroidal components of the magnetic field trapped in the rings are directly generated by induction. The rings act simply as the secondary turn of the transformer whose primary turn is the conical coil itself. The toroidal components of the trapped field in the rings are generated by Hall currents. The gradients of the magnetic field produced by the conical theta-pinch coil effectively produce a charge separation that drives poloidal currents. These, in turn, generate the toroidal magnetic field[46].

If the double-ended conical theta pinch is now cut in half and the two ends reversed in position in the magnetic guide field, as shown in

FIGURE 3



Production of corotational and contrarotational plasma vortex structures in a conical theta pinch.

Figure 3, the result will be a pair of corotating and contrarotating ring structures approaching each other. Theory predicts that two corotational or two contrarotational rings will superimpose to form one stronger ring while a corotational and contrarotational ring will interact. This interaction is the result of the fundamental nonlinear character of the partial differential equations describing the flow.

In 1963, Wells investigated these structures; first, by propagating them along a magnetic solenoid and then, by placing the halves of the two-ended conical theta pinch inside a magnetic mirror that acted as the guide field replacing the solenoid. The conical theta-pinch guns were placed inside the mirror field at each conjugate point and then energized by a capacitor bank. When the circuit was closed on each pinch coil, the current underwent damped oscillations.

A plasma ring was produced on each half-cycle of the current; but every other ring was not in equilibrium in the guide field and broke up, producing a hot surrounding plasma for the alternate rings to move through. Each remaining ring at each end of the mirror was either a corotational or contrarotational structure. Thus, the series of rings produced by each conical theta pinch consists of a series of either corotational or contrarotational structures.

Because of the gradient in the magnetic field of the mirror, the rings formed first slow down. Then the following rings catch up to them, coalesce with them, and form one strong corotational and one strong contrarotational ring that meet at the center of the mirror. It might be expected that the interaction of the corotational and contrarotational structures would destroy them, forming a turbulent fireball. Fortunately, this is not what happens; instead the interaction takes the form of magnetic barrier that holds the rings apart while they slowly decay, thermalize, and eventually break up.

All these phenomena have been carefully observed in the laboratory over a period of 14 years. The rings from each end of the mirror do indeed coalesce, and the coalesced rings that meet at the center of the mirror form a magnetic barrier or "septum." Ion temperatures in these rings in the currently operating machine are approximately 350 eV. The electron temperature is very low ($Te \approx 30eV$) and the density is typically 10^{16} particles/cm^3. The septum and the rings are observable for about 30 microseconds after they meet at the center of the mirror. These rings have been compressed with a secondary magnetic field to produce ion temperatures as high as 6 Kev. (This is discussed further in a later section.)

Plasma Paravortices and Orthovortices

In normal fluids, vortices are formed when a transverse velocity shear occurs. They appear to serve as natural "roller bearings" and alleviate the viscous stresses, resulting in an equilibrium configuration. In a plasma, however, vortex formation is a much more complex process; since both hydrodynamic (Magnus) forces, and electromagnetic (Lorentz) interactions are involved. As can be seen in Figure 4, the Lorentz force and the fluid dynamic force have opposite signs, and this sign difference is what gives plasma vortex behavior its complexity. (See Figures 4, 5, and 6.)

Putting aside differences in simple geometry, only one type of hydrodynamic vortex structure exists; and by virtue of the definition of

FIGURE 4

THE SIMILARITY BETWEEEN ELECTROMAGNETIC
AND FLUID DYNAMIC EQUATIONS

Electrodynamics	Fluid dynamics
$\frac{\partial H_i}{\partial x_j} - \frac{\partial H_j}{\partial x_i} = J_k$	$\frac{\partial V_i}{\partial x_j} - \frac{\partial V_j}{\partial x_i} = \zeta_k$
$\frac{\partial H_i}{\partial x_i} = 0$	$\frac{\partial V_i}{\partial x_i} = 0$
$J_i H_j - J_j H_i = \frac{f_k^{Lor.}}{\mu_0}$	$\zeta_i V_j - \zeta_j V_i = \frac{-f_k^{Mag.}}{\zeta}$

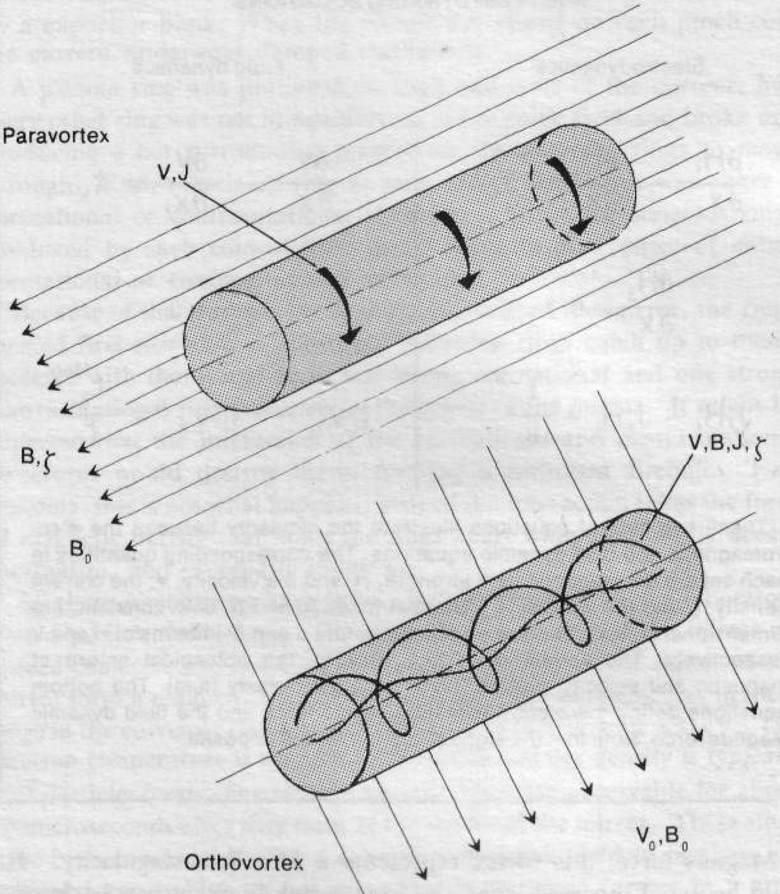
These two sets of equations illustrate the similarity between the electromagnetic and fluid dynamic equations. The corresponding quantities in each set are the magnetic field strength, H, and the velocity, V; the current density, j, and the vorticity ζ. The quantities μ_0 and p, both constant, are dimensional factors. The top equations define J and ζ in terms of H and V respectively. The middle equations express the solenoidal nature of magnetic and velocity vectors (for a constant density fluid). The bottom equations define the electrodynamic Lorentz force and the fluid dynamic Magnus force. Note that the signs of the forces are opposite.

the Magnus force, this vortex represents a force-free singularity. It should be noted, however, that a plasma vortex filament can be force-free in two ways: (1) the Lorentz force and Magnus force both can vanish; or (2) they may be both finite but self-canceling.

If there is a violent electromagnetic disturbance, two types of vortices will be formed. One, the paravortex, has its axis aligned with the magnetic field and generally moves normal to the local magnetic field lines. The other, elucidated by Busemann[21] and discussed above, is more complicated and has been termed an orthovortex, moving parallel to the magnetic field lines. Figure 6 schematically describes these two types of plasma vortices.

Paravortices have been observed in a number of laboratory plasmas, and are formed as the result of a Rayleigh-Taylor instability when plasma is accelerated across a magnetic field. Under the influence of an external force, the charged particles drift at right angles to both the force and the magnetic field. The positive ions will dominate one side of the

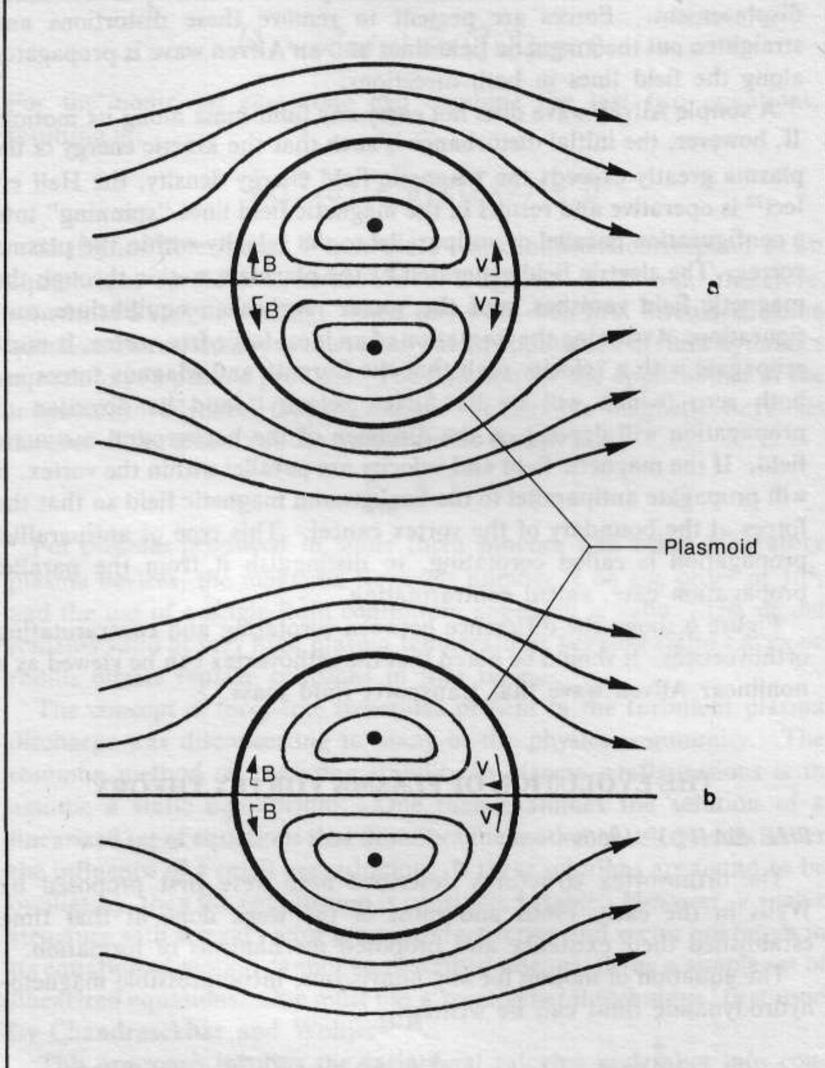
FIGURE 5
PARAVORTEX AND ORTHOVORTEX FILAMENTS



The paravortex and orthovortex filaments depicted here show the pattern of their constituent fields. The paravortex has its velocity and current density in the azimuthal direction. The vorticity vector and the magnetic field are axial and are aligned with the background magnetic field. The velocity profile of the paravortex is nearly that of a rigid rotator. The orthovortex has all four of its vector fields aligned. These fields are spiral with variable pitch as a function of radius. The axis of the orthovortex filament is perpendicular to the background magnetic field.

"ripples" and will be deficient on the other side. This charge separation generates an electric field that in turn causes the ripples to increase in magnitude until they finally break into a pair of paravortices.

FIGURE 6



a Contrarotating vortex traveling to the right
b Corotating vortex traveling to the left.

The orthovortex is more complicated, but more symmetric than the paravortex. In the case of the orthovortex, the magnetic field is deformed by the plasma as it moves across it, resulting ultimately in a background field everywhere parallel or antiparallel to the velocity vector of the plasma. Suppose a column of plasma in a uniform plasma background is displaced perpendicular to some initially imposed magnetic field. Given a plasma with high conductivity, the magnetic field will be "frozen" into the plasma and deformed by the displacement. Forces are present to remove these distortions and straighten out the magnetic field lines and an Alfven wave is propagated along the field lines in both directions.

A simple Alfven wave does not carry any fiuid mass along its motion. If, however, the initial disturbance is such that the kinetic energy of the plasma greatly exceeds the magnetic field energy density, the Hall effect[22] is operative and results in the magnetic field lines "spinning" into a configuration parallel or antiparallel to the velocity within the plasma vortex. The electric field generated by the plasma's motion through the magnetic field vanishes, and the vortex reaches an equilibrium configuration. Assuming the formation of an ideal force-free vortex, it must propagate with a velocity such that the Lorentz and Magnus forces are both zero (which will be its Alfven velocity), and its direction of propagation will depend on the direction of the background magnetic field. If the magnetic field and velocity are parallel within the vortex, it will propagate antiparallel to the background magnetic field so that the forces at the boundary of the vortex cancel. This type of antiparalllel propagation is called corotating, to distinguish it from the parallel propagation case, called contrarotating.

Figure 6 shows the difference between corotating and contrarotating orthovortices. It should be noted that the orthovortex can be viewed as a nonlinear Alfven wave that transports fluid mass.

THE EVOLUTION OF PLASMA VORTEX THEORY

THE EARLY 1960s

The orthovortex structures described here were first proposed by Wells in the early 1960s and most of the work done at that time established their existence and proposed mechanisms of formation.

The equation of motion for any nonviscous, incompressible magnetohydrodynamic fluid can be written:

$$\rho\frac{\partial\vec{v}}{\partial t} = -\vec{\nabla}\left(p + \frac{1}{2}\rho v^2\right) + (\vec{j}\times\vec{B}) - \rho(\vec{\xi}\times\vec{v})$$

where $\vec{\xi}\,(\equiv\vec{\nabla}\times\vec{v})$ is the vorticity vector, $\vec{v}$ = the velocity vector, t = time, ρ = the density, p = the pressure, $\vec{j}$ = the current density, and $\vec{B}$ = the background magnetic field. For a steady state condition, this equation can be reduced to:

$$\vec{\nabla}\times(\vec{j}\times\vec{B}) = \rho\,\vec{\nabla}\times(\vec{\xi}\times\vec{v}).$$

If one assumes a parallel flow as the qualitative theory of plasma orthovortices suggests, such that $\vec{v} = \alpha\vec{H}$ and $\vec{\xi} = \alpha\vec{j}$ where α is a scalar function of position, the equation further reduces to:

$$(1 - \alpha^2\rho/\mu)\ \vec{\nabla}\times(\vec{J}\times\vec{B}) = 0.$$

For the nontrivial case, one can combine the last two equations, resulting in:

$$\vec{\nabla}\times(\vec{J}\times\vec{B}) = \rho\vec{\nabla}\times(\vec{\xi}\times\vec{v}) = 0$$

Shafranov determined [23] that these two conditions correspond to an equilibrium toroidal structure in a magnetic field and, therefore, constitute a magnetovortex. When this work was first discussed, some question was raised about the validity of an application of fluid dynamics equations to a plasma problem. The criterion for the applicability of the magnetohydrodynamic approximation[24] is that the magnetic Reynolds number of the plasma under consideration is such that:

$$R_m \geq 1.$$

For plasmas produced in some theta pinches and other laboratory plasma devices, the magnetic Reynolds number is on the order of 10^4, and the use of a single-fluid continuum approach is valid. The model remains valid at very high plasma temperatures because finite Larmour radius effects replace collisions in that regime.

The concept of force-free structures present in the turbulent plasma discharge was disconcerting to many in the physics community. The common method of analyzing stability in plasma configurations is to assume a static equilibrium. One then examines the solution of a linearized set of equations that describes the motion of the systems under the influence of a small perturbation. If these solutions are found to be oscillatory, then the equilibrium is considered stable. However, a vortex structure with several vector cross-product terms and vector operators in its equation of motion cannot be described adequately by a simple set of linearized equations. One must use a more general technique, first used by Chandrasekhar and Woltjer[25-26].

This approach involves the variational calculus and takes into consideration not only small perturbations, but large mass motions as well. By considering the total energy of the vortex structure as a closed system and varying that energy over the volume of the plasmoid, one can determine the lowest free energy state, and hence the most stable one. The total energy, E, of the closed plasma configuration containing internal, magnetic, and kinetic energies is identical to the free energy in the case of an incompressible fluid we consider here for this introductory discussion. It can be written:

$$E = \int_V \left(\frac{1}{2}\rho v^2 + \frac{1}{2}\frac{B^2}{\mu_0} + U\right)dV$$

where U is the internal energy, V is the volume, and μ_0 the magnetic permeability. The energy can now be varied, subject to a set of constraint integrals imposed on the flow, by the method of Lagrange multipliers. A possible set of such constraint integrals derived directly from the equations of motion of a conducting fluid[26–29] is:

$$I_1 = \int \vec{A}\cdot\vec{B}\, dV$$

$$I_2 = \int \vec{B}\cdot\vec{v}\, dV$$

$$I_3 = \int \vec{A}\cdot\vec{v}\, dV$$

$$I_4 = \int \hat{i}\times\vec{r}\cdot\rho\vec{v}\, dV$$

$$I_5 = \int \hat{j}\times\vec{r}\cdot\rho\vec{v}\, dV$$

$$I_6 = \int \hat{k}\times\vec{r}\cdot\rho\vec{v}\, dV$$

$$I_7 = \int \rho\, dV$$

If the plasma is assumed to be a constant density, an incompressible fluid that can transfer angular momentum to its surroundings, one need impose only constraints I_1 and I_2. The variation problem can now be written:

$$\delta E - \frac{\alpha}{2\mu_0}\,\delta I_1 - \rho\beta\delta I_2 = 0$$

where $\frac{\alpha}{2\mu_0}$ and $\rho\beta$ are the Lagrange multipliers. Expanding and simplifying this equation gives one:

$$\int_V \frac{1}{\mu_0}\,\vec{\nabla}\times\vec{A} - \frac{\alpha}{2\mu_0}\,\vec{A} - \rho\beta\vec{v})\cdot\vec{\nabla}\times\delta\vec{A}\,dV =$$

$$\int_V \left(\frac{1}{\mu_0}\vec{\nabla}\times\vec{\nabla}\times\vec{A} - \frac{\alpha}{2\mu_0}\,\nabla\times A - \rho\beta\vec{\nabla}\times\vec{\nabla}\right)\cdot\delta A\; dV +$$

$$\int_S \left(\frac{1}{\mu_0}\,\vec{\nabla}\times\vec{A} - \frac{\alpha}{2\mu_0}\,\vec{A} - \rho\beta\vec{r}\right)\times\delta\vec{A}\cdot d\vec{S}$$

At this point, Woltjer argues that the surface terms are zero because he assumes a rigid boundary. That assumption, however, is not necessarily applicable to a plasma vortex, and it must be assumed that $\vec{A}$ is arbitrary both within and on the plasmoid. Since the variations $\vec{A}$ and $\vec{v}$ are independent, their coefficients must vanish separately, and one finds on the interior:

$$\vec{\nabla}\times\vec{B} = \alpha\vec{B} + \rho\mu_0\beta(\vec{\nabla}\times\vec{v})$$

$$\vec{v} = \beta\vec{B}$$

and at the boundary:

$$\frac{1}{\mu_0}(\vec{\nabla}\times\vec{A}) - \frac{\alpha}{2\mu_0}\vec{A} - \rho\beta\vec{v} = 0 .$$

The energy, expressed in terms of I_2 then is:

$$E = (\rho\beta + \frac{1}{\mu_0}\beta)\frac{I_2}{2} .$$

Minimizing this with respect to β, one finds:

$$\beta^2 = \frac{1}{\mu_0\rho} .$$

Note that this corresponds to the equipartition solution, since now

$$\frac{\rho v^2}{2} \equiv \frac{B^2}{2\mu_0}.$$

This also mathematically confirms the qualitative assumption made earlier that a stable solution corresponds to collinear flow.

From the solution of the variational equations, one determines:

$$\vec{v} = \frac{\vec{B}}{\rho\beta\mu_0} - \frac{\alpha\vec{A}}{2\mu_0\rho\beta} .$$

It is apparent that as $\alpha\beta$ approaches zero, β approaches $\pm(\mu_0\rho)^{-\frac{1}{2}}$ at the same rate. Since β is finite,

$$\vec{\nabla}\times\vec{B} = \frac{\alpha}{(1-\rho\mu_0\beta^2)}\vec{B} = K\vec{B},$$

and the field is force-free.

By the end of the early 1960s there was a preliminary picture of a stable plasmoid with collinear flows having Alfven velocity. Both the Lorentz and Magnus forces are zero, and the fields $\vec{j}$, ξ, $\vec{r}$, and $\vec{B}$ are all mutually parallel.

THE LATE 1960s

Most of the theoretical work during the mid and late 1960s concerned a refinement and elaboration of the early theory of vortex formation. This work was encouraged by the growing number of experimental reports (discussed in a later section of this article) that confirmed many of the features of the plasma vortex theory. One of the major accomplishments was a theoretical proof of vortex existence without the use of variational calculus.

For a three-dimensional analysis of a plasma in a solenoidal vector field, it is useful to employ a stream function, ψ, which is most easily defined in cylindrical polar coordinates: ω, Θ, and z. For an axially symmetric flow, these equations take the form:

$$v_z = \frac{1}{\omega}\frac{\partial\psi}{\partial\omega}$$

$$v_\omega = -\frac{1}{\omega}\frac{\partial\psi}{\partial z}$$

$$v_0 = \frac{\Omega}{\omega}$$

$$\frac{\partial}{\partial t}(\Delta^*\psi) + \frac{2\Omega}{\omega^2}\frac{\partial\Omega}{\partial z} + \frac{1}{\omega}\frac{\partial(\psi_1\Delta^*\psi)}{\partial(\omega,z)} + \frac{2}{\omega^2}\frac{\partial\psi}{\partial z}\Delta^*\psi = \nu\Delta^{*2}\psi$$

$$\frac{\partial\Omega}{\partial t} \quad \frac{1}{\omega}\frac{\partial(\psi,\Omega)}{\partial(\omega,z)} \quad \nu\Delta^*\Omega$$

and finally,

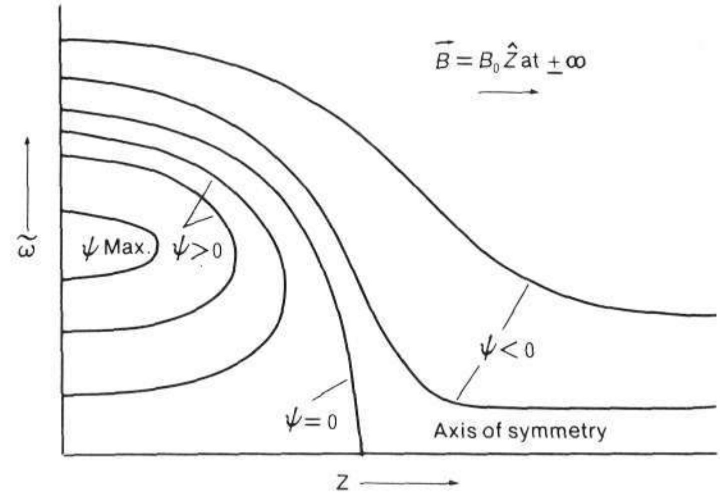
$$\frac{\partial(\psi,\Delta^*\psi)}{\partial(\omega,z)} = \frac{\partial\psi}{\partial\omega}\frac{\partial(\Delta^*\psi)}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial(\Delta^*\psi)}{\partial\omega} .$$

The stream function satisfying these equations has been determined [30] to be:

$$\psi = C\sin^2\beta\cosh^2\eta\sinh^2\eta_0[{}^-\sinh^2\eta + (\cos^2\beta/\sinh^2\eta_0)(\sinh^2\eta_0 {}^- \sinh^2\eta)],$$

This stream function describes a spheroidal vortex in oblate spheroidal coordinates and is schematically depicted in Figure 7. Chandrasekhar[43]

FIGURE 7



Stream functions for force-free plasmoid.

and others[25] demonstrated that any axisymmetric solenoidal vector field can be written as a superposition of poloidal and toroidal fields in terms of four scalar fields, P,T,U,and V.

Letting

$$\vec{h} \equiv \vec{H}/(\ \pi\rho)^{1/2}$$

where ρ is the mass density of the fluid, the fields can be written:

$$\vec{h} = -\ \omega \frac{\partial P}{\partial z}\hat{\omega} + \omega T\hat{\theta} + \frac{1}{\omega}\frac{\partial}{\partial\omega}(\omega^2 T)\hat{z}$$

$$\vec{v} = -\ \omega \frac{\partial U}{\partial z}\dot{\omega} + \omega V\hat{\theta} + \frac{1}{\omega}\frac{\partial}{\partial\omega}(\omega^2 U)\hat{z}\ .$$

From this it can be seen that the vorticity of an axisymmetric solenoidal vector field exhibits a certain reciprocity. The poloidal field generates a toroidal vorticity, and the toroidal field generates a poloidal vorticity. Going back to the MHD approximation, described above, an incompressible inviscid fluid of finite conductivity can be discussed by:

$$-\ \frac{\partial\vec{h}}{\partial t} = \vec{\nabla}x\left(\frac{1}{4\pi\nabla}\ \nabla x\vec{h} - \vec{v}x\vec{h}\right)$$

and

$$\cdot\ \frac{\partial\vec{v}}{\partial t} + (\vec{v}\vec{\nabla})\vec{v} - \frac{1}{4\pi\rho}\ \nabla x\vec{h}x\vec{h} = -\ \vec{\nabla}\left(\frac{P}{\rho} + x\right)$$

where x is the gravitational potential and ρ is the pressure. These two equations can be transformed into four simultaneous differential equations involving scalars P,T,U, and V, mentioned above. After considerable simplification, and assuming the case of infinite conductivity, a static set of equations can be generated:

$$0 = \{\omega^2 U, \omega^2 P\}$$

$$0 = \{\omega^2 U, T\} + \{V, \omega^2 P\}$$

$$0 = \{\omega^2 T, \omega^2 P\} + \{\omega^2\}$$

where

$$\{\phi, \psi\} \equiv \frac{\partial\phi}{\partial z}\frac{\partial\psi}{\partial\omega} - \frac{\partial\phi}{\partial\omega}\frac{\partial\psi}{\partial z} \equiv \frac{\partial(\phi,\psi)}{\partial(\omega,z)} .$$

One solution of these equations if P = U and T = V, which corresponds to the collinear or parallel flow of $\vec{h}$ and $\vec{v}$, and $\vec{j}$ and ξ. These correspond to the solutions obtained using variational calculus in the work of the early 1960s.

If the stream function is $\psi = 0$ at the outer boundary of the plasmoid, it can be shown[31] that the net azimuthal current in the plasma must be opposite in direction to the azimuthal current in the surrounding solenoidal magnet windings. Complete analysis indicates that at their core, the vortex rings have purely toroidal flow — they flow around the center of the ring. In order to satisfy boundary conditions, the surface of the plasmoid has no toroidal flow component at all, and the velocity, current density, and magnetic and vortex fields all circle the ring cross section as poloidal fields. These are the corotational and contrarotational orthovortices discussed earlier in this report.

THE EARLY 1970s

With the basic theory established during the previous decade, work in the early 1970s shifted emphasis to the application of the elementary theory to controlled thermonuclear fusion. The inherent stability of the plasmoids lent itself well to possible confinement schemes, and one in particular seemed to show exceptional promise.

In the late 1960s, investigators observed experimentally that when two plasmoids were directed at each other from opposite ends of a solenoidal guidefield, they would meet and remain stationary for a fraction of a millisecond. One of the major goals during this period was the development of a theoretical understanding of the phenomena consistent with the elementary principles of a force-free plasmoid.

When the pinch coils are allowed to ring (that is, they are not crowbarred or "shorted out"), several plasmoids are generated. Half were found to survive while the other half, whose internal current orientations were such that they produced magnetic fields parallel to the guide field, were destroyed. Although the first plasmoid has the largest trapped fields, the later plasmoids were observed to travel faster and to overtake it. These plasmoids coalesce and travel along the axis of the solenoidal guide field across the vacuum chamber until they meet coalesced plasmoid from the other end of the machine. The first step in understanding the plasmoid collision is an explanation of this process of coalescing or superposability. A general system of equations describing an isochoric, conducting fluid may be written as:

$$\frac{\partial \vec{v}_i}{\partial t} + (\vec{v}_i \cdot \vec{\nabla})\vec{v}_i = - \nabla(P/\rho) + \frac{\mu_o}{\rho}\vec{J}_i x \vec{H}_i + \upsilon \nabla^2 \vec{v}_i$$

$$\vec{\nabla} \cdot \vec{v}_i = 0$$

$$\vec{\nabla} x \vec{H}_i = \vec{J}_i$$

$$\vec{\nabla} x \vec{E}_i = - \frac{\partial(\mu_o \vec{H}_i)}{\partial t}$$

$$\vec{J}_i = \nabla(\vec{E}_i + \mu_o \vec{v}_i \; x \; \vec{H}_i)$$

$$\vec{\nabla} \cdot \vec{J}_i = 0$$

where $\vec{E}$ represents the electrical field strength, υ the viscosity, $\vec{v}$ the velocity of the fluid, $\vec{H}$ the magnetic field intensity, $\vec{p}$ the pressure , ϱ the density of the fluid, μ the magnetic permeability, j the current density, and t the time. The subscript i is used to denote the different magnetofluids. Each vector field, as well as additive hydromagnetic motions (that is, $\mathbf{E}_1 + \mathbf{E}_2$) must satisfy the above equations. The criteria for additivity of the different fields are found by setting the nonlinear terms in the above expressions equal to zero. This may be written for two magnetofluids as:

$$\vec{\nabla} x(\vec{\xi}_2 x \vec{V}_1 + \vec{\xi}_1 x \vec{V}_2) = \frac{\mu_o}{\rho} \vec{\nabla} x(\vec{J}_2 x \vec{H}_1 + \vec{J}_1 x \vec{H}_2)$$

and

$$\vec{V}_1 x \vec{H}_2 + \vec{V}_2 x \vec{H}_1 = 0$$

where ξ_i is the vorticity of $\vec{V}_i$ ($\vec{\nabla} x \vec{V}_i \equiv \vec{\xi}_i$). These equations can now be used to represent self-superposable motions by equating subscripts, such that:

$$\vec{\nabla}\times(\vec{\xi}\times\vec{V}) = \frac{\mu_0}{\rho}\,\vec{\nabla}\times(\vec{J}\times\vec{H})$$

$$\vec{V}\times\vec{H} = 0 \quad .$$

The concept of self-superposable flow is extremely important in discussing fluid motions. Self-superposability is required in order that the fluids decay linearly in the presence of dissipative terms; and without it, as the fields begin to decrease, nonlinear effects would change the flow patterns and the nonsteady behavior would become extremely complicated. An example of this is the anomalous diffusion observed in many experiments. Linear calculations are made to determine the growth rate of plasma fluting, ballooning, and so forth. These growth rates are not observed experimentally, because as the plasma starts to move it leaves the limits of the linear approximation and there is a nonlinear, turbulent diffusion.

The equations above show that collinearity is a sufficient condition for self-superposable flow. Of course, this is just the condition for lowest energy state obtained by the variational calculation. Thus, force-free, collinear plasmoids should be observed even in the presence of dissipative mechanisms. In fact, dissipation always exists in real plasmas.

Some investigators have expressed doubt that a steady Beltrami flow can exist in a viscous medium, but because of the self-additive property, the time dependence does not change the Beltrami (that is, force-free) nature of the fluid. The plasma can begin and end its life as a force-free, collinear entity.

Note that two separate flows are superposable when:

$$\vec{v}_1 = \sqrt{\mu_0/\rho}\;\vec{H}_1 \quad \text{and} \quad \vec{v}_2 = \sqrt{\mu_0/\rho}\;\vec{H}_2 \ ,$$

or

$$v_1 = -\sqrt{\mu_0/\rho}\;\vec{H}_1 \quad \text{and} \quad v_2 = -\sqrt{\mu_0/\rho}\;\vec{H}_2 \quad .$$

These are the equations representing either two contrarotating or two corotating plasmoids. Plasmoids produced by the same gun have exactly this property, while plasmoids that collide are created by different guns and are of opposite "parity." The colliding plasmoids will not be simply additive because they satisfy the condition,

$$\vec{v}_1 = \pm\,\sqrt{\mu_0/\rho}\;\vec{H}_1 \quad \text{and} \quad v_2 = \pm\,\sqrt{\mu_0/\rho}\;\vec{H}_2 \quad .$$

It can be shown[45] that: (1) any axial symmetric flow of a conducting fluid is always self additive when in an axially symmetric field; and (2) two axially symmetric flows of an infinitely conducting fluid are superposable if the velocity is parallel to the magnetic field in each of the two flows. Early researchers apparently failed to realize that this proof also allowed the flows both to be antiparallel.

Now, consider two of these coalesced plasma vortex structures colliding, with one corotating and the other contrarotating. Each plasmoid constitutes collinear flow, and assuming a common guide field they will move toward each other maintaining their form and structure. In a plasmoid radius of R there is no observed interaction until the separation distance is 2R. What happens at that point appears to be extremely complicated, and investigators have developed only a qualitative theory based on experimental observations. As the vortex structures get closer together, the two collinear flow fields (velocity and magnetic) mix to an increasing degree. The plasmoids become smaller and increase their speed. During the initial period of interaction, the plasmoids retain their structure, and the toroidal currents cause each vortex ring to create and maintain its own magnetic field. These fields interact as the rings approach and form a "magnetic wall" that becomes better defined as the plasmoids approach each other. This wall prevents the motion of charged particles from one ring to another, but neutral particles can mix.

Experiments have confirmed the various phases of this qualitative picture of plasmoid collision many times, but there has been little headway in the mathematical modeling of the process. Numerical calculations of the $\vec{J} \times \vec{B}$ and $\varrho(\xi \times \vec{V})$ interactions between the two plasmoids have been carried out on a digital computer[32]. These show that the colliding corotating and contrarotating plasma vortices exhibit a weak dipole-dipole attraction at large distances with a much stronger short range repulsion as they approach the point of collision.

Another recent theoretical endeavor is a rigorous reexamination of the variational approach to plasma vortex stability using Clebsch potentials, undertaken by Wells and Rund [33]. It is hoped that this elegant mathematical revision of Chanresakhar and Woltjer's original work for a two-species magnetofluid will once and for all provide a firm, vigorous theoretical foundation for the concept of magnetovortices in plasma physics.

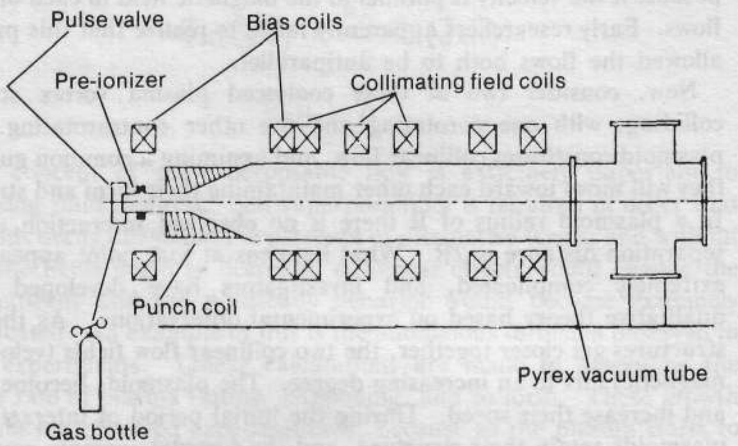
THE EXPERIMENTAL WORK ON PLASMA VORTICES

THE EARLY 1960s

The early 1960s saw the birth of the TRISOPS experimental program. Encouraged by the development of the theory explaining mass plasma rotations as a vortex phenomenon, Dr. Wells at the Princeton Plasma Physics Laboratory , designed and built a specific apparatus (TRISOPS I) to further test the theory.

FIGURE 8

TRISOPS I



TRISOPS I (see Figure 8) consisted of a conical theta pinch located at one end of a 1.5 meter drift tube. The theta pinch was a cone-shaped, single-turn coil with a quarter-cycle rise time of 2.5 microseconds and a maximum operating voltage of 25 kV. It could be crowbarred so that the current did not reverse. The entire drift tube was placed in a solenoidal magnetic field of 4,000 gauss. Gas was introduced by means of a pulsed gas valve, and the apparatus could be run in either static or dynamic modes. Plasma diagnostics consisted of microwave measurements of density and mass motion, streak photographs, and electric and magnetic probe measurements.

Plasma densities on the order of 10^{15} particles per cubic centimeter were produced when the pinch coil was allowed to ring. Streak and framing photographs showed an initially filamentary structure followed by a plasma vortex ring produced at each subsequent half-cycle of the pinch coil. A velocity of 7.5×10^6 cm/sec was measured for the first plasmoid produced, which photographically appeared to be the strongest. Examination of the plasma structure a short time after the first plasmoid was formed (after several, weaker rings had also been produced) showed a coalescing of plasma structure into one long plasmoid with a well-formed head and a long, tenuous tail. This was the first indication of superposability, explained theoretically several years later.

Magnetic probe measurements indicated that the plasma rings had a trapped poloidal magnetic field of 1,000 gauss and a trapped toroidal magnetic field of 200 gauss. This observation of trapped magnetic fields was the first concrete piece of evidence in support of the vortex theory of force-free plasmoids. Electric probe measurements of the radial field at the wall indicated that the plasma as a whole carried a net positive charge. It was shown further that inside the plasma, $\vec{E}r$ was not proportional to the radius, indicating that the rotation could not be considered to be rigid body rotation. The theory also predicted non-conservation of angular momentum, adding further credence to the concept of vortex structures.

Immediately following this work, investigators changed various parameters to determine their effect on plasma vortex formation. The conical theta-pinch coil was crowbarred (that is, prevented from ringing), and no rotational motion was observed. Crowbarring also lowered the plasma density by three orders of magnitude compared to the noncrowbarred case, and reduced the radial electric field effectively to zero. Variations in the strength of the guide field showed than an optimum field strength existed for a given theta-pinch coil voltage, and that any large variation of magnetic guide field on this value prevented plasma vortex formation.

THE MID AND LATE 1960s

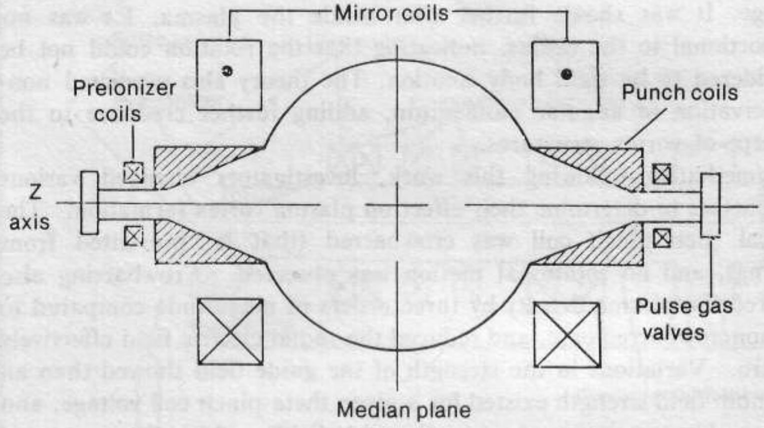
By 1965, all usable applications of the single-ended drift tube just described had been explored, and a second generation TRISOPS II was designed and built[9]. Its initial construction was completed at the Princeton Plasma Physics Laboratory, but most of the data was collected at the University of Miami after Wells, the principal investigator, joined the physics faculty there in 1965.

TRISOPS II, described schematically in Figure 9, was built primarily to study the interaction of two plasma vortex structures when they are fired toward each other inside a magnetic bottle or guide field. The machine consists of two water-cooled coils that could be operated either in a mirror or cusp field configuration. The conical theta-pinch guns were placed directly under the coils so that they fired into a negative magnetic field gradient. The vacuum chamber consisted of an 18-inch Pyrex sphere especially built to accommodate the theta-pinch coils. The total stored energy that could be delivered to the pinch coil was 10,000 J. The field generated by the theta pinch is shown in Figure 10. The mirror field had no shear windings and a ratio of 1.25:1.

Observations were made on five separate modes of operation of machine. With the mirror coils inoperative and a static gas fill, the plasma moved into the spherical chamber guided only by the fringing field of the pinch coils. It has been shown theoretically that a background field was essential to stable vortex formation; and in the absence of this field, it was observed experimentally that the plasmoids broke up immediately after leaving the theta-pinch gun and filled the entire chamber with nonstructured plasma.

With a static gas fill and mirror guide field, well-defined plasma

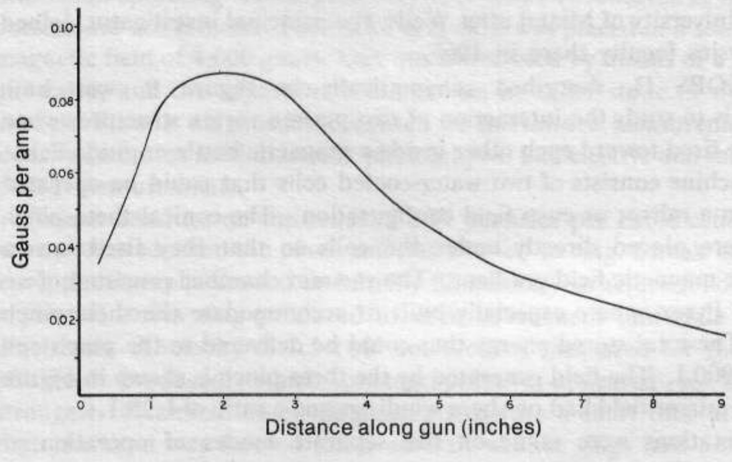
FIGURE 9



A schematic drawing of the TRISOPS II experiment.

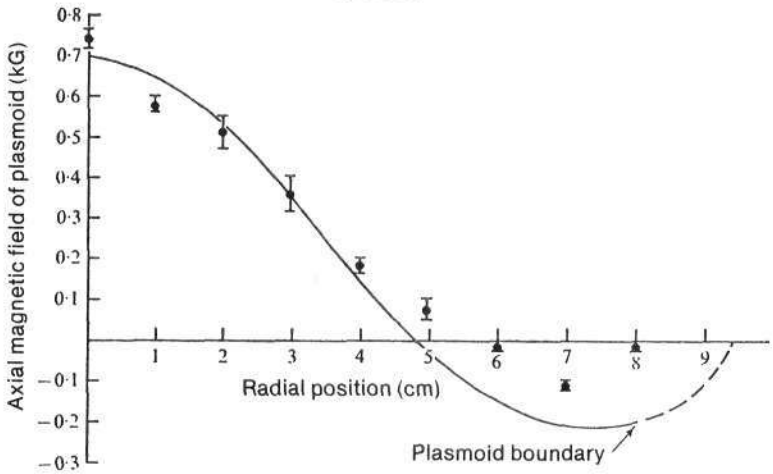
FIGURE 10

MAGNETIC FIELD IN THE CONICAL THETA PINCH OF TRISOPS II



vortex structures were formed. The plasmoids produced on alternate half cycles of each of the pinch coils coalesced, and the two composite plasmoids moved into the chamber as observed in TRISOPS I. The translational velocity of structures increased and the diameters decreased as they approached the center of the chamber. At the point of impact, the plasmoids maintained their integrity and remained separated by a dark septum for 200 microseconds. Then the plasmoids

FIGURE 11



A comparison of a theoretical curve with experimental data from TRISOPS II for the plasmoid's axial magnetic field. Deuterium is at 35μ; main bank 12 kV; preionizer 18 kV; guide-field 0.8 kG.

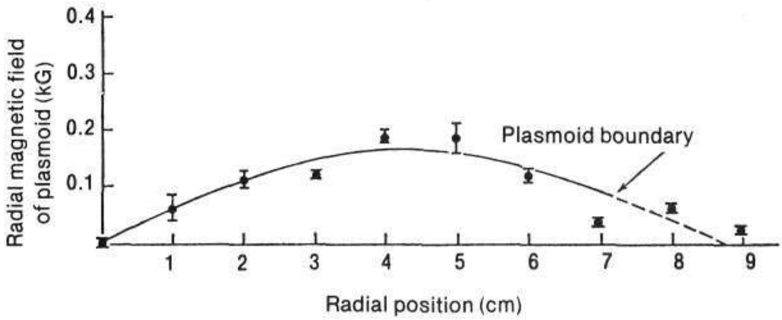
increased in diameter and started to move back toward their source mirrors, indicating a magnetic field reversal within the structures. Magnetic probe measurements experimentally confirmed this internal field reversal, and ion temperatures of 350 eV were measured.

The theory of corotating and contrarotating plasma orthovortices formation and superposability predicted that for a given guide field, only every other plasmoid would be in equilibrium and the remainder would be destroyed. This implies that for the coalesced rings, the magnetic moment should be additive, an implication confirmed experimentally with magnetic probes. As the coalesced vortices come close together at the center of the mirror, the opposing circulation patterns of the two highly conducting plasmoids (one corotating and the other contrarotating) should pile up the magnetic field and result in a magnetic barrier or wall between the two structures. Magnetic probes confirmed the existence of this wall, and the wall was observed visually on the streak photographs as a dark septum separating the two structures.

A third mode of operation in the TRISOPS II was pulsed gas valves with mirror field. There was little difference between the results obtained in this mode and the static-fill mode described above, except the plasmoids appeared to have a slightly higher velocity and density. The final two modes were pulsed and had static gas fills with cusped field. *In both cases, the operation of the guide coils in a cusp geometry resulted in the breakup of all plasmoid structure.*

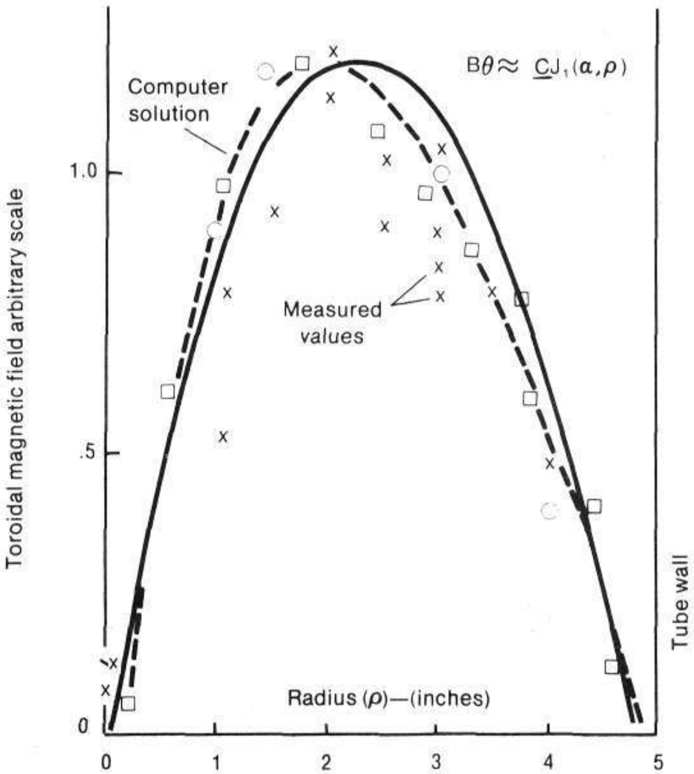
The theory had progressed far enough by this time to predict numerical values for different plasmoid field components, and an effort was made to determine if these values could be confirmed experimentally[34]. To this end, researchers made extensive use of magnetic probe measurements, and the results exceeded all expectations. As can be seen in Figures 11, 12, and 13, the experimental data agreed extremely well with the theoretically predicted curves.

FIGURE12



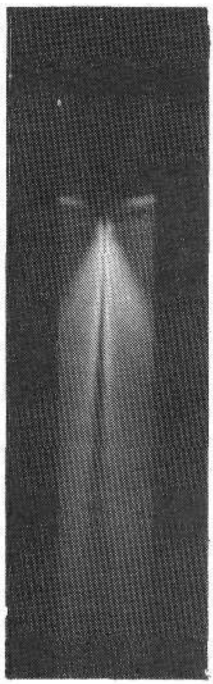
A comparison of the theoretical curve with experimental data from TRISOPS II for the plasmoid's radial magnetic field. Deuterium is at 35 ; main bank at 12 kV; preionizer 18kV; guide-field 0.8 kG.

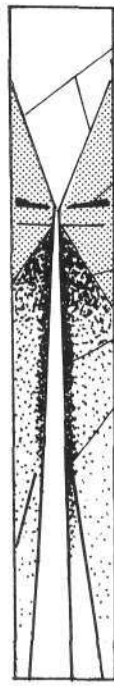
FIGURE 13



A toroidal magnetic field versus radius.

FIGURE 14





This streak picture and schematic diagram illustrate the collision of vortex structures at the center of a secondary mirror system in TRISOPS III. Time goes from top to bottom of the picture and diagram. The streak time is 50μ sec. The horizontal slit was placed along the centerline of the machine, between the secondary compression coils.

THE 1970s

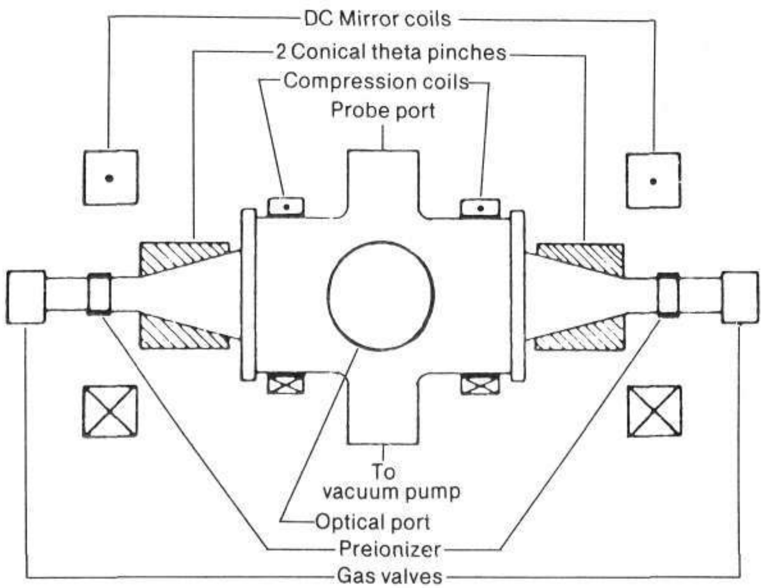
Studies of colliding plasmoids continued into the early and mid 1970s using TRISOPS II[13], followed by two more thermonuclear test machines, TRISOPS III and IV[14].

Work with TRISOPS II consisted of an experimental study of the role of the guide field on vortex formation and an investigation into the nature of the septum or magnetic wall formed when the two plasmoids collide.

A magnetic probe registered a diamagnetic signal when the plasmoids arrived, indicating the formation of a magnetic wall in the external guide field. This well is not to be confused with the magnetic wall formed *between* the two colliding plasmoids. Although the magnitude of this well increased with both guide-field intensity and theta-pinch gun voltage, the percentage reduction of guide field levels off with a maximum field reduction of 85 percent. It is now believed that the localized field reduction generated by internal currents of the plasma structures accounts for the relatively long time during which the plasmoids remain stable and stationary.

Studies of the dark septum visible on streak photos (see Figure 14) reconfirmed the existence of the magnetic barrier between colliding

FIGURE 15



A schematic diagram of the TRISOPS III machine used for secondary compression of plasma vortex structures. The primary-mirror strength was 900 G.

plasmoids. It was determined that there is no superposition of the corotational and contrarotating vortices, and no azimuthal current in the septum region was detected. Magnetic probes showed a weak azimuthal field in the septum, along with relatively strong radial and axial fields.

TRISOPS III

It is obvious to anyone working on the fusion problem that any apparatus that can produce stable, force-free structures that remain stationary for periods of over 100 microseconds holds definite promise for a thermonuclear application. Using the plasmoid's self-generated stability and magnetic well confinements, all that is necessary is to heat the vortex structures to temperatures high enough to sustain a thermonuclear reaction. TRISOPS III was designed and built to investigate this concept[14].

In TRISOPS III (see Figure 15) a pair of theta-pinch coils are located at each end of a primary magnetic mirror. The plasmoids are guided by the steady-state mirror field to the center of the machine. There they collide and are compressed by a secondary mirror system located at the center of the primary mirror system. Since the current flow in these secondary compression coils is in the same direction as currents flowing in the primary mirror coils, by Lentz's Law the plasmoids are compressed and their currents are amplified. Deuterium gas is introduced into the vacuum system (base pressure of 3×10^{-6} torr) through pulsed gas valves.

The conical theta pinches were powered by a single 1 μF, 50 KV capacitor, with a quarter cycle rise time of .5 microseconds and with the capacitors charged at 18 KV. This discharge was sufficient to produce a 20 KG peak magnetic field in the throat of the theta-pinch guns. The secondary compression coils were powered by a quarter megajoule capacitor bank whose construction and operation are described in detail elsewhere[35].

Since the rings were stationary in the laboratory frame, investigators found that there was no coupling problem and no apparent limit to the size of currents that could be produced without causing instabilities in the ring. The compression mirror acted as its own magnetic containment bottle, and currents of 120 KA were induced in toroidal plasmoids compressed to a 2 cm major diameter. Thermalization began when the compression field became high enough to overcome the short-range repulsive forces that held the plasmoids apart. This decay of plasma structure resulted in the formation of a normal unstructured mirror plasma. All temperatures referred to in this article were obtained at peak secondary compression before the rings thermalized and are, therefore, characteristic of the ordered plasma vortex

A multichannel Fabry-Perot interferometer[36] was especially developed to measure ion temperature by observing Doppler Broadening, and ion temperatures of 170 eV have been observed at 14.3 KV on the 20 KV compression capacitor bank. Ion density was measured by scanning the D_β line at 4861 A°. Peak density at 14.3 KV on the compression bank was 1×10^6 ions/cm^3.

It was observed further that the plasma trapped in the vortex structures remained in the system for 20 microseconds, a time considerably greater than that predicted for classical mirror diffusion. Electron temperatures at 8 KV on the compression bank were measured by taking line-intensity ratios of He II at 4686° and He I at 5875°. At peak compression, the electron temperature was approximately 10 eV, with a corresponding ion temperature at 30 eV. For operation of TRISOPS III at a mirror ratio of 1.4, the $\eta\tau$ was equal to 10^{12} at 14.3 KV on the compression bank.

To summarize the results with TRISOPS III: it demonstrated the feasibility of using the compression of stable vortex structures to generate plasmas of some thermonuclear interest. The entire time history of plasmoid generation, collision, and compression can best be seen in Figure 14, a streak photo of a shot with TRISOPS III.

TRISOPS IV

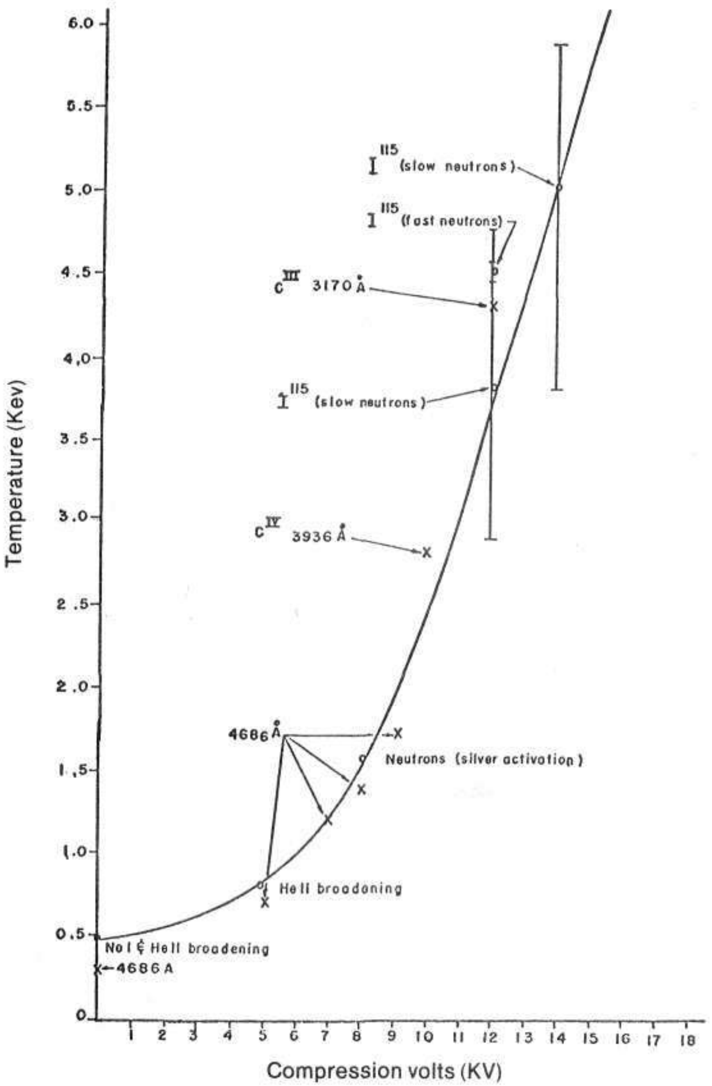
TRISOPS IV, the present thermonuclear machine, is essentially a refinement of TRISOPS III with the major difference that the current machine is constructed around a 3-inch vacuum chamber. This allowed the secondary compression coils to be smaller in diameter, resulting in a greater "squeeze" upon the plasmoids although the same compression bank was used as in TRISOPS III. Another refinement was the development of more effective preionizers.

The development of diagnostic methods was directed toward measuring ion and electron temperature and density. Most of the data were collected using a static deuterium gas fill of 300 microns. The laboratory staff developed an eight-channel photoelectric monochromator[37] that greatly expedited ion temperature measurements using Doppler Broadening. The device splits an 8A°-wide line from a Jarrall-Ash Ebert monochrometer into eight components, using eight fiber optic ribbons. Each ribbon activates a separate photo-multiplier tube, resulting, with only one shot, in an accurate line profile within an 8A° window. The device also proved invaluable in measuring Stark profiles for density measurement. Measurement of neutron flux produced by the compression of plasmoids also played a major role in the diagnostic effect. Determination of electron temperature was attempted by measuring X-ray fluxes due to Bremsstrahllung radiation.

The principal effort has been directed toward the measurement of ion temperatures over the whole range of operation from no compression to 14 KV bank voltage (which corresponds to 38.69 KG at the center of the mirror). For low compression voltages, broadening of the He II 4686 A° line was used. At higher compression fields, this line tended to be swamped by the continuum, and carbon III and IV lines (3170 A° and 3936 A° respectively) were examined. Temperatures ranged from 480 eV uncompressed to 4.0 keV at 12 KV compression. These data, along with data obtained by neutron flux measurement, are presented in Figure 16. The relation between the compression bank voltage and the magnetic field at the center of the mirror is shown in Figure 17.

Considerable time was also devoted to making density measurements using Stark broadening. This method, suggested by Einar Hinnov[38], worked well for lower compression voltages, but ran into difficulties at higher voltages. At low temperatures, the broadening of the two helium II lines was quite practical, but as the ion temperature approached 2 keV, the width became too broad to be accurately examined by the eight-channel device's 8 A° window. To remedy the situation, the laboratory staff obtained a new grating for the Ebert monochrometer, widening the window width of the spectrometer to 16 A°. This allowed density measurement up to 12 KV bank compression (corresponding to a field of 37.2 KG and ion temperature of 4.0 KeV). The data obtained are presented in Figure 18.

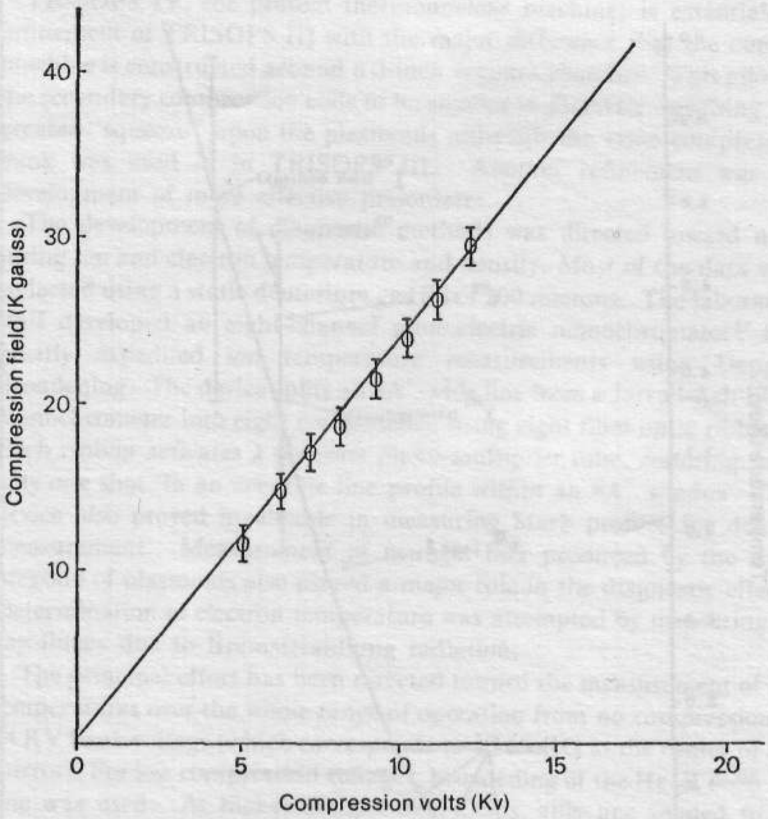
FIGURE 16



Temperature of plasma (in Kev) plotted against compression voltage in TRISOPS IV. Diagnostic methods for each point are indicated by the spectral lines used (or neutron activation targets).

Accurate determination of plasmoid density, along with a knowledge of plasmoid size (see Figure 19) and containment time, not only provided data that are essential in deriving the $\eta\tau$ of the TRISOPS IV; it also

FIGURE 17



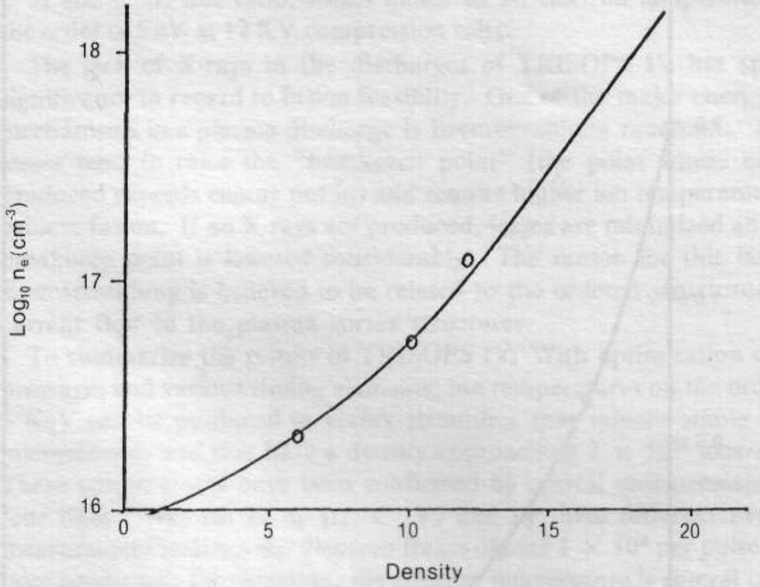
Compression field in TRISOPS IV in K Gauss plotted against compression voltage.

enabled the analysis of neutron flux data to provide a cross-check for temperature measurement.

Neutron flux was measured in three ways. For lower compression voltages (up to 8 KV), specially constructed "neutron counters" designed and built by the Los Alamos Laboratory[39], were used. These measured the induced beta activity in Ag (n, β) Ag. The electromagnetic disturbances accompanying a high bank voltage discharge interferred with the operation of the geiger tubes incorporated in the counter and made them very unreliable for high-temperature neutron flux measurements.

At higher compression voltages, two other techniques were used. One involved the moderation of the fusion neutrons by a paraffin block, and a measurement of the thermal-neutron-induced resonance activity in In(n, γ) In. Since the spectrum of neutron energies leaving a paraffin block of

FIGURE 18



Logarithim (base 10) of plasma density (in particles/cm) plotted against compression voltage for plasmoids in TRISOPS IV.

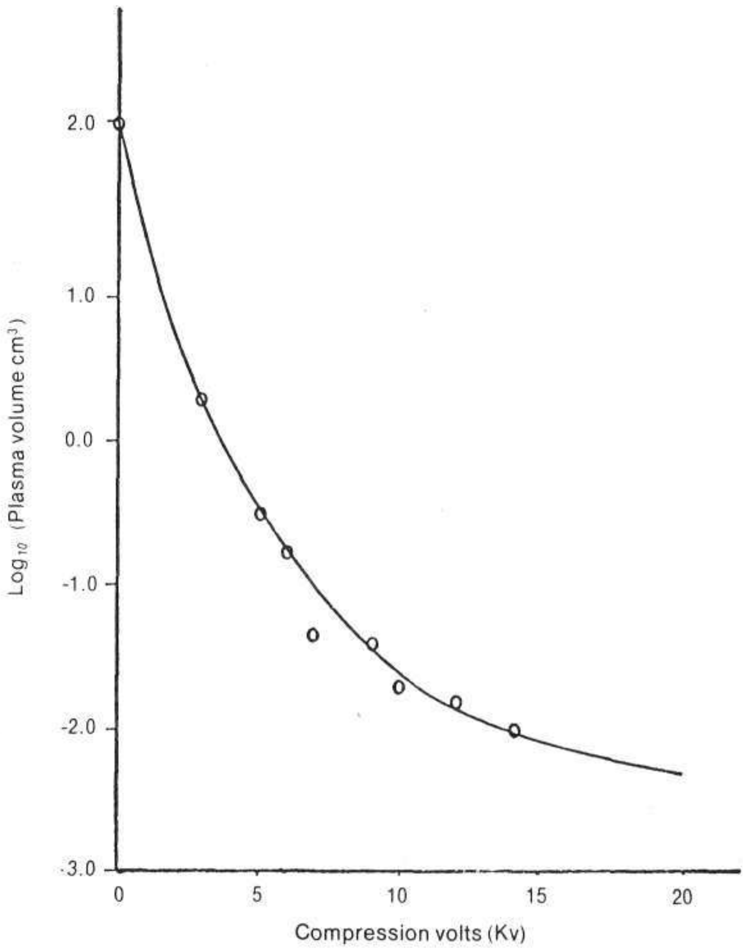
a given thickness is known[40], this worked very well, except for the difficulty that only a small portion of the incident neutrons is actually moderated to thermal energies.

The second technique involved direct use of the 2.4 MeV D-D fusion neutrons in the In(n, nγ)In reaction. When two or more techniques were applied, for a given bank voltage the results were identical within experimental error, and for all but extremely high compression voltages, all neutron data agreed with optical measurement of temperature. Neutron yields of approximately 1.3 $\times$ 10^8 have been measured for 14 KV compression volts (fields of 39 KG, indicating an ion temperature of 4.9 KeV. Using the fast neutron In(n,$\eta\gamma$)In reaction, a test for isotropicity was made; and for all exposed angles (about 20 percent of the area surrounding the discharge is unavailable for foil placement because of vacuum ports, optical windows, and so forth), the neutron flux was isotropic.

Time-resolved studies are not yet complete, nor is the effort for an accurate determination of neutron energy that would show conclusively the origin of the neutrons.

Electron temperatures in the TRISOPS IV discharge are very low, on the order of 5 eV. This was determined by two methods. The flux of soft

FIGURE 19



Logarithim of plasmoid volume (in cm^3) plotted against compression voltage in TRISOPS IV.

X-rays due to Bremsstrahlung is dependent on electron energy, and by measuring this flux one can determine electron temperature. A photographic darkening technique was used in an attempt to measure the X-ray flux, but no fogging was detected although the most sensitive X-ray film available was used. After several reruns of the experiment, and after confirming that the film was good (by exposure to a calibrated soft X-ray source), this null result was used to set an upper limit on the electron temperature of 300 eV at 14 KV compression volts (corresponding to a field of 39KG and an ion temperature of 4.9 KeV).

The second method used to determine electron temperature was a C II and C III line ratio, which indicated an electron temperature on the order of 5 eV at 12 KV compression volts.

The lack of X-rays in the discharges of TRISOPS IV has special significance in regard to fusion feasiblity. One of the major energy loss mechanisms in a plasma discharge is Bremsstrahlung radiation. These losses tend to raise the "breakeven point" (the point where energy produced exceeds energy put in) and require higher ion temperatures to achieve fusion. If no X-rays are produced, losses are minimized and the breakeven point is lowered considerably. The reason for this lack of Bremsstrahlung is believed to be related to the ordered structures and current flow in the plasma vortex structures.

To summarize the results of TRISOPS IV: With optimization of fill pressures and various timing elements, ion temperatures on the order of 5 KeV can be produced in vortex structures that remain stable for 8 microseconds and that have a density approaching 2×10^{17} ions/cm^3. These temperatures have been confirmed by optical measurements on four lines (Ne, He II, C III, C IV) and by three different neutron measurement techniques. Neutron fluxes of over 1×10^8 per pulse have been produced. Furthermore, the electron temperature is several orders of magnitude lower than ion temperature, and radiative losses due to Bremsstrahlung X-ray emission are almost nonexistent.

THE FUTURE

As stated earlier, TRISOPS IV is the device in current use, but because essentially all the useful data have already been collected from it, plans are underway for a new apparatus to be built early next year. TRISOPS V will involve a modification of TRISOPS IV so that the compression magnetic field (now persisting for about 8 microseconds) will remain on for a considerable longer time. To accomplish this, a new "transmission-line"-type capacitor bank power supply is under construction that will allow a field on time of about 100 microseconds, with a peak compression field of approximately 100,000 gauss. The vacuum chamber and theta pinches also will be completely rebuilt and enlarged so that plasmoid size at peak compression can be increased from .02 cm^3 to 3.0 cm^3. To accomplish this, the theta-pinch coils will be lengthened to ionize a larger volume of gas, and a more tapered chamber will be built to allow the larger rings to propagate without riding on the walls of the vacuum vessel as they do in the present machine. The conical theta pinches each will be powered by a small Marx bank in order to attain higher preignition ion temperatures in the vortex rings.

The purpose of the new design is to find out if it is possible to heat a large plasmoid to several KeV with densities of 2×10^{17} while holding it for times on the order of 100 microseconds.

If the effort is completed (full funding has not yet been obtained) and proves to be successful, the next phase planned is the construction of prototype reactor that will actually generate power. Preliminary studies, based on information obtained with TRISOPS III and IV have been completed, and the results look very promising.

The design parameters for the prototype reactor are taken in part from a Los Alamos study[41] and are based on preliminary but reasonable data. The basic design consists of a magnetic mirror 24 inches long, 2.5 inches in diameter at the conjugate points (throat), and 6 inches at the center. The mirror coils will be a modification of the original Bitter magnet design[42], and the result will be a six-turn Bitter magnet contoured to form a magnetic mirror.

The energy for compression will be stored in a transmission line identical to that proposed for TRISOPS V, and will consist of "capacitor modules" that can be built up in stages as experience is gained with the machine. Initially there will be 24 modules, each module accomodating 16, 90 μfd, 10KV (25KA) capacitors. Given a 1.0 cm^3 plasmoid volume with a density of 1×10^{18} ions/cm^3 heated to 10 KeV, a containment time of 1 millisec will be needed for an $\eta\tau$ of 10^{15} that is necessary for net power generators. Assuming a pulsed mode of operation with a cycle time of 30 seconds, basic design analysis[41] predicts a net power output of 1.2 megawatts for a fusion reactor with the parameters described.

CONCLUSION

Although the power plant stage is a number of years in the future, the TRISOPS project suggests that the use of naturally occurring stability states may be the key to fusion energy. Combinations of temperature, density, and containment time comparable to or higher than any other reported results recently have been obtained by magnetically compressing plasma vortex structures, and preliminary observations indicate that TRISOPS is by no means near its limit of heating and containment. Future studies, particularly the development of TRISOPS V, will show whether plasma vortices hold the answer to the world's energy problems.

References

1. N. Rostoker and A. Kolk. 1961. *Phys. Rev. 124: 965.*
2. *G. Schmidt. 1962. Phys. Fluids* 5: 636.
3. B. Taylor, private communication.
4. H. Badin, et al. 1962. *Nuclear Fusion Suppl.* Part 2: 521.
5. W.I. Linlor. 1961. *Nucl. Fusion 1: 309.*
6. *J.W. Dungey. 1962. Nucl. Fusion 1: 312.*
7. *D.R. Wells and G. Schmidt. 1963. Phys. Fluids 6: 418.*
8. *D.R. Wells. 1962. Phys. Fluids 5: 1016.*

9. D.R. Wells 1966. *Phys. Fluids* 9: 1010.
10. J. Norwood and D.R. Wells. 1969. University of Miami Plasma Physics Report MIAPH-PL-69.15.
11. D.R. Wells and J. Norwood. 1069. *J. Plasma Physics* 3: 21.
12. D.R. Wells. 1970. *J. Plasma Phys.* 4:645.
13. E. Nolting, P. Jindra, and D.R. Wells. 1973. *J. Plsma Phys.* 9: 1.
14. D.R. Wells, et al., 1974. *Phys. Rev. Letters* 3⅜: 1203.
15. W.H. Bostick, W. Prior, and E. Farver. 1965. *Phys. Fluids* 8: 745.
16. L. Hogberg and K. Vogel. 1961. *Nucl. Instr. Methods* 10: 95.
17. R.W. Waniek. 1960. In *Proceedings of the 10th International Astronautical Congress.* Vienna. p. 131.
10. V.S.Komelkov, et al. 1962. In *Proceedings of the 5th International Conference of Ionization Phenomena in Gases.* p. 2191.
19. L. Konen, et al. 1968. *Conference on Plasma Physics and Controlled Nuclear Fusion Research.* Novosibrisk. CN-24/K-4.
20. W.B. Jones and R.D. Miller. 1967. General Electric Research and Development Center Report 67-C-178.
21. A. Busemann. 1961. Air Force Office of Scientific Research Lecture in Honor of the 80th Anniversary of Theodor von Karman. NASA Langley Research Center, Langley Field, Virginia.
22. J. Norwood and D.R. Wells. (1969) University of Miami Plasma Physics Report MIAPH-PL-69.15: 6.
23. V.C. Shafranov. 1957. *Zhur. Eksptl. i Teoret. Fiz.* 33: 3.
24. W.B. Thompson. 1962. *An Introduction to Plasma Physics.* New York: Pergammon Press, Inc.
25. S. Chandrasekhar. 1956. *Proc. Natl. Acad. Sci.* (U.S.) 42: 1.
26. L. Woltjer. 1959. *Proc. Nat. Acad. Sci.* (U.S.) 45: 769.
27. L. Woltjer. 1958. *Proc. Nat. Acad. Sci.* (U.S.) 44: 489.
28. L. Woltjer. 1958. *Proc. Nat. Acad. Sci.* (U.S.) 44: 833.
29. L. Woltjer. 1961. *Rev. Modern Phys. 32: 914.*
30. V. O'Brien. 1961. *Quart. J. Appl. Math.* 19: 163.
31. K.E. Wakefield, private communication.
32. J. Davidson, to be published.
33. D.R. Wells, H. Rund, and L. Hawkins, to be published in *Jour. of Plas. Phys.*
34. D.R. Wells, 1964. *Phys. Fluids* 7:826.
35. M.S. Spiegel. 1974. *The generation of vortex rings in a conical theta pinch,* University of Miami PhD Dissertation.
36. F.N. Cooke. 1974. University of Miami Physics Department. PhD Dissertation.
37. L. Phadke, paper to be presented at the 19th Annual Meeting of the American Physical Society Plasma Physics Division, Atlanta, Georgia, November 1977.
38. E. Hinnov, private communication.
39. R. Lanter and D. Bannerman. 1966. Los Alamos Report LA-3498-MS.
40. F.R. Jones. 1950. *Shields for Neutron Sources.* AEC Report HW-21169.
41. Scaling laws for the linear theta pinch: A comparison of magnetic and laser heating. LA-5434-MS.
42. D.R. Wells. 1956. Princeton Laboratory Report TM-30.
43. S. Chandrasekhar. 1961. *Hyrdrodynamic and Hydromagnetic Stability.* Oxford, England: University Press.
44. S.K. Trehan. 1958. *Astrophys. J.* 127: 446.
45. E. Nolting. 1971. AFOSR Scientific Report AFOSR-TR-71-2905.
46. W.H. Bostick and D.R. Wells. 1963. *Phys. Fluids* 6: 1325.

H. L. Helmholtz.

On Integrals of the Hydrodynamic Equations That Correspond to Vortex Motions

BY HERMANN VON HELMHOLTZ

Hitherto, integrals of the hydrodynamical equations have been sought almost exclusively under the assumption that the orthogonal components of the velocity of each water particle can be set equal to the differential quotients in the corresponding directions of a certain func-

TRANSLATOR'S NOTE

Helmholtz's paper on vortex motions was first published in 1858 in the Journal fur die reine and angewandte Mathematik*. *It was the first in a series of ground-breaking papers in hydrodynamics published by Helmholtz in the decade between 1858 and 1868. In exemplary fashion it expresses his attempt at applying hydrodynamic considerations to electromagnetic phenomena, and electrodynamic models to the mathematical explication of complex hydrodynamical situations. The full force of this mode of thinking is being recognized only now when Helmholtz's basic ideas on vortex motion have found fruitful application in the investigation of high energy plasmas by researchers like Wells [this issue] and Bostick [IJFE, March 1977].*

A rough translation of Helmholtz's 1858 paper was presented by P.G. Tait in the Philosophical Magazine and Journal of Science**. *In a short postscript of his translation Tait wrote: "The above version of one of the most important recent investigations in mathematical physics was made long ago for my own use, and does not pretend to be an exact translation." The translation here is therefore the first precise rendering into English of Helmholtz's original and is presented in order to make readily available to the contemporary physicist the paper that originates the precise mathematical treatment of the concepts of vortex lines and vortex filaments that play an increasingly important role in plasma physics.*

*Vol. 55, pp. 25-55.
**Supplement to Vol. 33, Fourth Series (18??)

tion, which we shall call the velocity potential. Indeed, Lagrange has shown that this assumption is permissible whenever the motion of the water mass has been produced by and continued under the influence of forces that themselves can be represented as differential quotients of a force potential, and also that the influence of moving solids that come in contact with the fluid does not change the validity of that assumption. And, since most of the natural forces that can be well defined mathematically are representable as differential quotients of a force potential, by far the greater number of mathematically investigable cases of fluid motion fall into the category in which a velocity potential exists.

Yet, already Euler has pointed out that there are in fact cases of fluid motions in which no velocity potential exists; for instance, the rotation of a fluid about an axis with the same angular velocity for every particle. Among the forces that can produce such types of motion are magnetic forces acting upon a fluid conducting electric currents, and in particular friction, whether among the fluid particles or against fixed bodies. The influence of friction on fluids has not hitherto been mathematically definable; yet it is very great, except in the case of infinitely small oscillations, and it produces the most marked deviations between theory and reality. The difficulty of defining this effect and of finding methods for its measurement mainly consisted in the fact that no conception existed of the forms of motion that friction produces in fluids. In this regard it appeared to me to be of importance to investigate those forms of motion for which no velocity potential exists.

The following investigation will demonstrate that when there is a velocity potential, the smallest water particles have no rotational velocity, while at least a portion of the water particles is in rotation when there is no velocity potential.

By *vortex lines* I denote lines drawn through the fluid mass so that their direction at every point coincides with the direction of the momentary axis of rotation of the water particles lying on it.

By *vortex filaments* I denote portions of the fluid mass cut out from it by way of constructing corresponding vortex lines through all points of the circumference of an infinitely small surface element.

The investigation shows that if all the forces that act on the fluid have a potential: (1) no water particle that was not originally in rotation is made to rotate; (2) the water particles that at any given time belong to the same vortex line, however they may be translated, will continue to belong to the same vortex line; (3) the product of the cross section and the velocity of rotation of an infinitely thin vortex filament is constant along the entire length of the filament and retains the same value during all displacements of the filament. Hence the vortex filaments must run back into themselves in the interior of the fluid or else must end at the bounding surface of the fluid.

This last theorem enables us to determine the velocity of rotation when the form of the vortex filament at different times is given. Furthermore, the problem is solved of finding the velocities of the water particles for a given point in time if the velocities of rotation for this point in time are given; an arbitrary function, however, remains undetermined, and is to be applied to satisfy the boundary conditions.

This last problem leads to a peculiar analogy between the vortex motions of water and the electromagnetic effects of electric currents. Thus, if in a simply connected space* filled with a moving fluid there is a velocity potential, then the velocities of the water particles are equal to and in the same direction as the forces exerted on a magnetic particle in the interior of the space by a certain distribution of magnetic masses on its surface. If, on the other hand, vortex filaments exist in such a space, then the velocities of the water particles are to be set equal to the forces exerted on a magnetic particle by closed electric currents that in part flow through the vortex filaments in the interior of the mass, in part in its surface, and whose intensity is proportional to the product of the cross section of the vortex filaments and their velocity of rotation.

In the following I shall therefore frequently avail myself of the fiction of the presence of magnetic masses or of electric currents, simply in order to obtain a briefer and more vivid representation of the nature of functions that are the same kind of functions of the coordinates as the potential functions or attractive forces that attach to those masses or currents with respect to a magnetic particle.

By means of these theorems a series of forms of motion, concealed in the class of the unexamined integrals of the hydrodynamic equations, at least becomes accessible to the imagination even if the complete integration is possible only in a few of the simplest cases — as when we have one or two straight or circular vortex filaments in a mass of water that is either unlimited or partially bounded by an infinite plane.

It can be demonstrated that straight parallel vortex filaments, in a water mass limited only by planes perpendicular to the filaments, rotate about their common center of gravity, if for the determination of this point the velocity of rotation is considered analogously to the density of a mass. The position of the center of gravity remains unchanged. On the other hand, in the case of circular vortex filaments that are all perpendicular to a common axis, the center of gravity of their cross sections moves on a parallel to the axis.

1. Definition of Rotation

In a liquid capable of drop formation, at a point determined by the rectangular coordinates x, y, z, let p be the pressure at time, t; u, v, w

*I use this expression in the same sense in which Reimann (Journal fur die reine und angewandte Mathematik, Vol. 54, p. 108) speaks of simply and multiply connected surfaces.

the components of the velocity parallel to the coordinate axes; X, Y, and Z the components of external forces acting upon the unit of fluid mass; and h the density whose variations will be assumed to be vanishingly small. Then the known equations of motion for the interior points of the fluid are:

$$\left.\begin{aligned} X - \frac{1}{h}\cdot\frac{dp}{dx} &= \frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz} \\ Y - \frac{1}{h}\cdot\frac{dp}{dy} &= \frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz} \\ Z - \frac{1}{h}\cdot\frac{dp}{dz} &= \frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz} \\ 0 &= \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}. \end{aligned}\right\} \tag{1}$$

Hitherto, almost without exception, only such cases have been treated where the forces X, Y, Z, not only have a potential V so that

$$X = \frac{dV}{dx}, \quad Y = \frac{dV}{dy}, \quad Z = \frac{dV}{dz}, \tag{1a}$$

but also a velocity potential ϕ can be found so that

$$u = \frac{d\varphi}{dx}, \quad v = \frac{d\varphi}{dy}, \quad w = \frac{d\varphi}{dz}. \tag{1b}$$

Thereby the problem is immensely simplified, since the first three of the equations (1) give a common integral equation form which p is to be found ϕ having previously been determined so as to satisfy the fourth equation, which in this case takes on the form

$$\frac{d^2\varphi}{dx^2} + \frac{d^2\varphi}{dy^2} + \frac{d^2\varphi}{dz^2} = 0,$$

and thus coincides with the known differential equation for the potential of magnetic masses, which are external to the space for which this equation is assumed to hold. It is also known that every function φ that satisfies the above equation within a simply connected space can be

An n-fold connected space, accordingly, is one which can be cut by n-1 but no more surfaces without separating the space into two entirely separate parts. Thus a ring in this sense is a twofold connected space. The cutting surfaces all around must be bounded completely by the line in which they intersect the surface of the space.

expressed as the potential of a definite distribution of magnetic masses on the bounding surface, as I have already mentioned in the introduction.

In order that the substitution (1b) may be admissible, we must have

$$\frac{du}{dy}-\frac{dv}{dx}=0, \quad \frac{dv}{dz}-\frac{dw}{dy}=0, \quad \frac{dw}{dx}-\frac{dv}{dz}=0. \tag{1c}$$

To understand the mechanical meaning of these last three conditions, we may consider the change undergone by an arbitrary infinitely small volume of water during the time dt as composed of three different motions: (1) a translation of the water particle through space; (2) an expansion or contraction of the particle parallel to three main axes of dilatation so that every rectangular parallelepiped, made out of water, whose edges are parallel to the main directions of dilatation remains rectangular, while its edges may alter their length but remain parallel to their original direction; (3) a rotation about a temporary axis of rotation of arbitrary direction, which, according to a well-known theorem, may always be considered as the resultant of three rotations about the coordinate axes.

If the conditions (lc) are fulfilled at a point whose coordinates are ($\mathfrak{x}$, $\mathfrak{y}$, and $\mathfrak{z}$), then the values of u, v, w, and of their differential quotients at the point may be put as

$$\begin{aligned} u &= A, & \frac{du}{dx} &= a, & \frac{dw}{dy} &= \frac{dv}{dz} = \alpha, \\ v &= B, & \frac{dv}{dy} &= b, & \frac{du}{dz} &= \frac{dw}{dx} = \beta, \\ w &= C, & \frac{dw}{dz} &= c, & \frac{dv}{dx} &= \frac{du}{dy} = \gamma, \end{aligned}$$

whence we have for a point whose coordinates x, y, z, differ by an infinitely small quantity from ($\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$):

$$\begin{aligned} u &= A + a(x-\mathfrak{x}) + \gamma(y-\mathfrak{y}) + \beta(z-\mathfrak{z}), \\ v &= B + \gamma(x-\mathfrak{x}) + b(y-\mathfrak{y}) + \alpha(z-\mathfrak{z}), \\ w &= C + \beta(x-\mathfrak{x}) + \alpha(y-\mathfrak{y}) + c(z-\mathfrak{z}), \end{aligned}$$

or, if we let

$$\begin{aligned} \varphi = A(x-\mathfrak{x}) + B(y-\mathfrak{y}) + C(z-\mathfrak{z}) + \tfrac{1}{2}a(x-\mathfrak{x})^2 \\ + \tfrac{1}{2}b(y-\mathfrak{y})^2 + \tfrac{1}{2}c(z-\mathfrak{z})^2 + \alpha(y-\mathfrak{y})(z-\mathfrak{z}) + \beta(x-\mathfrak{x})(z-\mathfrak{z}) \\ + \gamma(x-\mathfrak{x})(y-\mathfrak{y}), \end{aligned}$$

then

$$u = \frac{d\varphi}{dx}, \quad v = \frac{d\varphi}{dy}, \quad w = \frac{d\varphi}{dz}.$$

It is known that by an appropriate choice of differently oriented coordinates x, y, z with origin at ($\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$) the expression for ϕ can be brought into the form

$$\varphi = A_1 x_1 + B_1 y_1 + C_1 z_1 + \tfrac{1}{2} a_1 x_1^2 + \tfrac{1}{2} b_1 y_1^2 + \tfrac{1}{2} c_1 z_1^2,$$

where the components $u, v, w,$ of the velocity with respect to the new coordinate axes have the values

$$u_1 = A_1 + a_1 x_1, \quad v_1 = B_1 + b_1 y_1, \quad w_1 = C_1 + c_1 z_1.$$

Velocity u_1 parallel to the x_1 axis is thus the same for all water particles for which x_1 has the same value, so that water particles that at the beginning of time dt are in a plane parallel to the $y_1 z_1$ plane will at the end of dt also lie in such a plane. The same holds for the $x_1 y_1$ *and the* $x_1 z_1$ plane. Thus if we imagine a parallelepiped bounded by three planes, parallel and infinitely close to the just mentioned coordinate planes, then the enclosed water particles after the passage of time dt will still form a rectangular parallelepiped whose surfaces are parallel to the same coordinate planes. Thus, the entire motion of such an infinitely small parallelepiped is, given assumptions (lc), composed of only a translation in space and a dilatation or contraction of its edges, but does not involve any rotation.

Let us return to the first coordinate system of $x, y, z,$ and suppose that aside from the motions of the infinitely small water mass surrounding point ($\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$) in existence so far, there exist additional rotational motions around axes through point ($\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$) and parallel to the $x, y,$ and z axes, whose angular velocities are ξ, η, ζ, Then the velocity components parallel to the coordinate axes of $x, y, z,$ contributed by these motions are:

$$\begin{array}{ccc}
0, & (z - \mathfrak{z})\,\xi, & -(y - \mathfrak{y})\,\xi, \\
-(z - \mathfrak{z})\,\eta, & 0, & (x - \mathfrak{x})\,\eta \\
(y - \mathfrak{y})\,\zeta, & -(x - \mathfrak{x})\,\zeta, & 0.
\end{array}$$

Thus the velocities of the particle with coordinates x, y, z now become:

$$\begin{aligned} u &= A + a(x-\mathfrak{x}) + (\gamma+\zeta)(y-\mathfrak{y}) + (\beta-\eta)(z-\mathfrak{z}), \\ v &= B + (\gamma-\zeta)(x-\mathfrak{x}) + b(\mathfrak{y}-y) + (\alpha+\xi)(z-\mathfrak{z}), \\ w &= C + (\beta+\eta)(x-\mathfrak{x}) + (\alpha-\xi)(y-\mathfrak{y}) + c(z-\mathfrak{z}). \end{aligned}$$

From these follows by differentiation:

$$\begin{cases} \dfrac{dv}{dz} - \dfrac{dw}{d\gamma} = 2\,\xi, \\ \dfrac{dw}{dx} - \dfrac{du}{dz} = 2\,\eta, \\ \dfrac{du}{dy} - \dfrac{dv}{dx} = 2\,\zeta. \end{cases} \tag{2}$$

The magnitudes on the left, therefore, which according to equations (1c) must be equal to zero, if a velocity potential is to exist, are equal to twice the rotational velocities of the water particles around the three coordinate axes. The existence of a velocity potential excludes the existence of rotational motions of the water particles.

As a further characteristic property of fluid motion with a velocity potential, we shall adduce here that no such motion can exist in a simply connected space, *S, which is completely filled with a fluid enclosed by* completely rigid walls. For if n denotes the normal of the surface of such a sphere, pointing to the interior, then the velocity component perpendicular to the wall $d\varphi/dn$ must be zero everywhere. Then, according to a well-known theorem by Green*:

$$\iiint \left[\left(\frac{d\varphi}{dx}\right)^2 + \left(\frac{d\varphi}{dy}\right)^2 + \left(\frac{d\varphi}{dz}\right)^2\right] dx\,dy\,dz = -\int \varphi\,\frac{d\varphi}{dn}\,d\omega,$$

where the integral on the left is to be extended over the entire space S, the integral on the right over the entire surface of S, a surface element of which is denoted by $d\omega$. Now, if $d\varphi/dn$ is equal to zero over the entire surface, then the integral on the left, too, must be equal to zero, which can be the case only if throughout the whole space S:

$$\frac{d\varphi}{dx} = \frac{d\varphi}{dy} = \frac{d\varphi}{dz} = 0,$$

thus, if no motion of the water takes place at all. Every motion of a bounded fluid mass in a simply connected space, when a velocity potential exists, therefore necessarily implies a motion of the fluid surface. If this surface motion, that is $d\varphi/dn$, is given in full, then as a result of this the entire motion of the enclosed fluid mass, too, is

* The previously noted theorem, which is not valid for multiply connected spaces.

uniquely determined. For if there were two functions $\varphi_{,}$ and $\varphi_{,,}$, which in the interior of space S were to simultaneously satisfy equation:

$$\frac{d^2\varphi}{dx^2}+\frac{d^2\varphi}{dy^2}+\frac{d^2\varphi}{dz^2}=0$$

and at the surface were to satisfy the condition:

$$\frac{d\varphi}{dn}=\psi$$

where ψ denotes the values of $d\varphi/dn$ determined by the given surface motion, then the function $(\varphi_{,}-\varphi_{,,})$, too, would satisfy the first condition in the interior of S, while at the surface we would have:

$$\frac{d(\varphi_{,}-\varphi_{,,})}{dn}=0$$

which, as just demonstrated, for the whole interior of S would imply:

$$\frac{d(\varphi_{,}-\varphi_{,,})}{dx}=\frac{d(\varphi_{,}-\varphi_{,,})}{dy}=\frac{d(\varphi_{,}-\varphi_{,,})}{dz}=0.$$

To both functions, therefore, exactly the same velocities also would correspond in the whole interior of S.

Thus rotations of the water particles and motions on a closed curve can occur in simply connected and entirely closed spaces only if no velocity potential exists. Therefore, in general, we may call motions without velocity potential vortex motions.

2. CONSTANCY OF VORTEX MOTION

First, we shall determine the variations of the rotational velocities ξ, η and ζ during the motion, when the only active forces are those that have a force potential. I first note in general that, if ψ is a function of x, y, z, and t and increases by $\partial\psi$ while the latter four magnitudes increase by ∂x, ∂y, ∂z, and ∂t, we have:
110-1.

$$\partial\psi=\frac{d\psi}{dt}\partial t+\frac{d\psi}{dx}\partial x+\frac{d\psi}{dy}\partial y+\frac{d\psi}{dz}\partial z.$$

If we now want to determine the change of ψt during the time interval ∂t for a water particle that remains constant, then we have to give to the magnitudes ∂x, ∂y , and ∂z the same values they have for the moving water particle, that is:

$$\partial x = u\partial t, \quad \partial y = v\partial t, \quad \partial z = w\partial t,$$

and we obtain:

$$\frac{\partial \psi}{\partial t} = \frac{d\psi}{dt} + u\frac{d\psi}{dx} + v\frac{d\psi}{dy} + w\frac{d\psi}{dz}.$$

In what follows I shall employ the symbol $\partial\psi/\partial t$ only in the sense that $(\partial\psi/\partial t)$ dt denotes the change of ψ during the time dt for the specific water particle, whose coordinates at the beginning of the time interval dt were x, y, and z.

If we elminate the magnitudes p by differentiation from the first of the equations (1) and simultaneously introduce the notation of equations (2), regarding equations (1a) as satisfiable for the forces X, Y, Z, we obtain the following three equations:

$$\left.\begin{aligned}\frac{\partial\xi}{\partial t} &= \xi\frac{du}{dx} + \eta\frac{du}{dy} + \zeta\frac{du}{dz},\\ \frac{\partial\eta}{\partial t} &= \xi\frac{dv}{dx} + \eta\frac{dv}{dy} + \zeta\frac{dv}{dz},\\ \frac{\partial\zeta}{\partial t} &= \xi\frac{dw}{dx} + \eta\frac{dw}{dy} + \zeta\frac{dw}{dz},\end{aligned}\right\} \tag{3}$$

or equivalently:

$$\left.\begin{aligned}\frac{\partial\xi}{\partial t} &= \xi\frac{du}{dx} + \eta\frac{dv}{dx} + \zeta\frac{dw}{dx},\\ \frac{\partial\eta}{\partial t} &= \xi\frac{du}{dy} + \eta\frac{dv}{dy} + \zeta\frac{dw}{dy},\\ \frac{\partial\zeta}{\partial t} &= \xi\frac{du}{dz} + \eta\frac{dv}{dz} + \zeta\frac{dw}{dz}.\end{aligned}\right\} \tag{3a}$$

If in a water particle ξ, η, and ζ simultaneously are equal to zero, then also:

$$\frac{\partial\xi}{\partial t} = \frac{\partial\eta}{\partial t} = \frac{\partial\zeta}{\partial t} = 0.$$

Hence, those water particles that do not already possess rotational motion do not attain such motion as time goes on.

As is well known, rotations can be composed according to the method of the parallelogram for forces. If ξ, η, ζ are the rotational velocities around the coordinate axes, then the rotational velocity q around the momentary axis of rotation is:

$$q = \sqrt{\xi^2 + \eta^2 + \zeta^2},$$

and the cosines of the angles of this axis with the coordinates are:

$$\xi/q, \ \eta/q \text{ und } \zeta/q.$$

If we now take in the direction of this momentary axis of rotation the infinitely small portion $q\varepsilon$, then the projections of this portion onto the three coordinate axis are $\varepsilon\xi$, $\varepsilon\eta$, $\varepsilon\zeta$. While at point x, y, z the components of the velocity are u, v, w, at the other endpoint of $q\varepsilon$ they are:

$$\begin{aligned} u_1 &= u + \varepsilon\xi \frac{du}{dx} + \varepsilon\eta \frac{du}{dy} + \varepsilon\zeta \frac{du}{dz}, \\ v_1 &= v + \varepsilon\xi \frac{dv}{dx} + \varepsilon\eta \frac{dv}{dy} + \varepsilon\zeta \frac{dv}{dz}, \\ w_1 &= w + \varepsilon\xi \frac{dw}{dx} + \varepsilon\eta \frac{dw}{ny} + \varepsilon\zeta \frac{dw}{dz}. \end{aligned}$$

At the end of time dt, therefore, the projections of the distance between the two particles, which at the beginning of dt limited the portion $q\varepsilon$, have attained values that, taking into account equations (3), may be written as follows:

$$\begin{aligned} \varepsilon\xi + (u_1 - u)\,dt &= \varepsilon\left(\xi + \frac{\partial \xi}{\partial t}\,dt\right), \\ \varepsilon\eta + (v_1 - v)\,dt &= \varepsilon\left(\eta + \frac{\partial \eta}{\partial t}\,dt\right), \\ \varepsilon\zeta + (w_1 - w)\,dt &= \varepsilon\left(\zeta + \frac{\partial \zeta}{\partial t}\,dt\right). \end{aligned}$$

The left-handed sides of these equations give the projections of the new position of the connecting line $q\varepsilon$, the right-hand sides the projection of the new velocity of rotation, multiplied by the constant factor ϵ; it follows from these equations that the connecting line between the two water particles, which at the beginning of time dt bounded the portion $q\varepsilon$ of the momentary axis of rotation, also after the lapse of time dt still coincides with the now-altered axis of rotation.

If we call *vortex line* a line whose direction coincides everywhere with the momentary axis of rotation of the water particles situated there, as we defined above, then the just-found theorem can be enunciated as follows: *Each vortex line remains continually composed of the same water particles, while it swims forward with these water particles in the fluid.*

The rectangular components of the velocity of rotation increase in the same proportion as the projections of the portion εq of the axis of rotation; from this it follows that *the magnitude of the resulting velocity of rotation in a specific water particle varies in the same proportion as the distance of this water particle from its neighbor in the axis of rotation.*

If we imagine that vortex lines are drawn through every point of the circumference of an infinitely small surface, then as a result of this a filament of infinitely small cross section is separated out from the fluid, which we shall call *vortex filament*. The volume of the portion of such a filament is bounded by two specific water particles, which, according to the just-proved theorems, is always filled by the same water particles, must remain constant during the motion, and its cross section, therefore, must vary inversely to its length. Hence the just-stated theorem also may be enunciated as follows: *The product of the velocity of rotation and the cross section in a portion of a vortex filament containing the same water particles remains constant during the motion of the filament.*

From equations (2) it follows immediately that:

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = 0 .$$

And, further, from this that:

$$\iiint \left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right) dx\, dy\, dz = 0 ,$$

where the integral may be extended over an entirely arbitrary portion S of the water mass. Through integration by parts it follows:

$$\iint \xi\, dy\, dz + \iint \eta\, dx\, dz + \iint \zeta\, dx\, dy = 0 ,$$

where the integrals are to be extended over the entire surface of the space S. Calling an element of this surface $d\omega$ and α, β, γ the three angles made with the coordinate axes by the normal $d\omega$, drawn outwards, we have:

$$dy\, dz = \cos\alpha\, d\omega , \quad dx\, dz = \cos\beta\, d\omega , \quad dx\, dy = \cos\gamma\, d\omega .$$

Hence:

$$\iint (\xi \cos\alpha + \eta \cos\beta + \zeta \cos\gamma)\, d\omega = 0 ,$$

or if σ is the resulting velocity of rotation, and ϑ the angle between its axis and the normal,

$$\iint \sigma \cos \vartheta \,.\, d\omega = 0\,,$$

the integral extending over the entire surface of S.

Now let S be a portion of a vortex filament, bounded by two infinitely small planes $\omega_{,}$ and $\omega_{,,}$ perpendicular to the axis of the filament. Then $\cos\vartheta$ is equal to 1 at one of these planes, equal to -1 at the other, and equal to 0 at the rest of the surface of the filament. Consequently, if $\sigma_{,}$ and $\sigma_{,,}$ are the rotational velocities in $\omega_{,}$ and $\omega_{,,}$ the last equation reduces to:

$$\sigma_{,}\,\omega_{,} = \sigma_{,,}\,\omega_{,,}\,,$$

from which it follows that *the product of the velocity of rotation and the cross section is constant throughout the entire length of a given vortex filament.* That it does not change as a result of the motion of the filament has been proven previously.

It also follows from this that a vortex filament can never end within the fluid, but must either return ring-shaped into itself within the fluid or reach to the boundaries of the fluid; for if a vortex filament ended anywhere within the fluid, a closed surface could be constructed for which the integral $\int \sigma \cos\vartheta$ would not have the value zero.

3. SPATIAL INTEGRATION

If the motion of the vortex filaments in the fluid can be determined, the stated theorems also will enable us to determine the magnitudes ξ, η, and ζ completely. We shall now consider the problem of finding the velocities u, v, and w from the magnitudes ξ, η, and ζ.

Thus, let there be given within a water mass that fills the space S the values of ξ, η, and ζ, which three magnitudes satisfy the condition that:

$$\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz} = 0\,. \tag{2a}$$

We want to find u, v, and w, so that within the entire space S they satisfy the conditions that:

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0\,, \tag{1}$$

$$\left.\begin{aligned} \frac{dv}{dz} - \frac{dw}{dy} &= 2\xi\,, \\ \frac{dw}{dx} - \frac{du}{dz} &= 2\eta\,, \\ \frac{du}{dy} - \frac{dv}{dx} &= 2\zeta\,. \end{aligned}\right\} \tag{2}$$

In addition there are the conditions determined by the particular nature of a given problem for the boundary of the space S.

For a given distribution of ξ, η, ζ we now may have vortex lines that within the space S are closed and return into themselves, as well as some that reach the boundary of S and there break off. If the latter is the case, one can continue these vortex lines, either on the surface of S or outside of S, and close them so that they return into themselves, so that a larger space S then comes into existence that contains only closed vortex lines, and for the entire surface of which ξ, η, ζ and their resultant σ are each equal to zero, or at least:

$$\xi \cos \alpha + \eta \cos \beta + \zeta \cos \gamma = \sigma \cos \vartheta = 0. \tag{2b}$$

As previously α, β, γ denote the angles between the normal of the portion of the surface of S under consideration and the coordinate axes, ϑ the angle between the normal and the resulting axis of rotation.

We obtain values of u, v, w, that satisfy the equations $(1)_4$ and (2) if we put:

$$\left.\begin{aligned} u &= \frac{dP}{dx} + \frac{dN}{dy} - \frac{dM}{dz}, \\ v &= \frac{dP}{dy} + \frac{dL}{dz} - \frac{dN}{dx}, \\ w &= \frac{dP}{dz} + \frac{dM}{dx} - \frac{dL}{dy}, \end{aligned}\right\} \tag{4}$$

and determine the magnitudes L, M, N, P by means of the conditions that within the space S :

$$\left.\begin{aligned} \frac{d^2 L}{dx^2} + \frac{d^2 L}{dy^2} + \frac{d^2 L}{dz^2} &= 2\xi, \\ \frac{d^2 M}{dx^2} + \frac{d^2 M}{dy^2} + \frac{d^2 M}{dz^2} &= 2\eta, \\ \frac{d^2 N}{dx^2} + \frac{d^2 N}{dy^2} + \frac{d^2 N}{dz^2} &= 2\zeta, \\ \frac{d^2 P}{dx^2} + \frac{d^2 P}{dy^2} + \frac{d^2 P}{dz^2} &= 0. \end{aligned}\right\} \tag{5}$$

The method of integrating these latter equations is known. $L, M, N,$ are the potential functions of imaginary magnetic masses distributed through the space S with the densities— $\xi/2\pi$, $-\eta/2\pi$, and $-\zeta/2\pi$ the potential function of masses which lie outside the space S. If we denote by r the distance of a point, whose coordinates are a, b, c, from the point $x, y, z,$ and by ξ_a, η_a, ζ_a the values of ξ, η, ζ at the point $a, b, c,$ then we have:

$$\left.\begin{aligned} L &= -\frac{1}{2\pi}\iiint\frac{\xi_a}{r}\,da\,db\,dc \\ M &= -\frac{1}{2\pi}\iiint\frac{\eta_a}{r}\,da\,db\,dc \\ N &= -\frac{1}{2\pi}\iiint\frac{\zeta_a}{r}\,da\,db\,dc, \end{aligned}\right\} \tag{5a}$$

the integration extending over the space S and:

$$P = \iiint\frac{k}{r}\,da\,db\,dc,$$

where k is an arbitrary function of $a, b, c,$ and the integration is to be extended over the exterior space enclosing S. The arbitrary function k must be taken so as to satisfy the boundary conditions, a problem whose difficulty is similar to that concerning electric and magnetic distribution.

That the values of $u, v,$ and w, given in (4), satisfy condition (1) is proved by differentiation, taking into account the fourth of equations (5).

We further find by differentiation of equations (4), taking into account the first three of equations (5), that:

$$\begin{aligned} \frac{dv}{dz} - \frac{dw}{dy} &= 2\xi - \frac{d}{dx}\left[\frac{dL}{dx} + \frac{dM}{dy} + \frac{dN}{dz}\right] \\ \frac{dw}{dx} - \frac{du}{dz} &= 2\eta - \frac{d}{dy}\left[\frac{dL}{dx} + \frac{dM}{dy} + \frac{dN}{dz}\right] \\ \frac{du}{dy} = \frac{dv}{dx} &= 2\zeta - \frac{d}{dz}\left[\frac{dL}{dx} + \frac{dM}{dy} + \frac{dN}{dz}\right]. \end{aligned}$$

Equations (2) are thus also satisfied, if it can be shown that in the entire space S :

$$\frac{dL}{dx} + \frac{dM}{dy} + \frac{dN}{dz} = 0. \tag{5b}$$

That this is the case follows from equations (5a):

$$\frac{dL}{dx} = +\frac{1}{2\pi}\iiint\frac{\xi_a(x-a)}{r^3}\,da\,db\,dc,$$

or, after integration by parts:

$$\frac{dL}{dx} = \frac{1}{2\pi}\iint \frac{\xi_a}{r}\, db\, dc - \frac{1}{2\pi}\iiint \frac{1}{r}\cdot\frac{d\xi_a}{da}\, da\, db\, dc,$$

$$\frac{dM}{dy} = \frac{1}{2\pi}\iint \frac{\eta_a}{r}\, da\, dc - \frac{1}{2\pi}\iiint \frac{1}{r}\cdot\frac{d\eta_a}{db}\, da\, db\, dc,$$
$$\frac{dN}{dz} = \frac{1}{2\pi}\iint \frac{\zeta_a}{r}\, da\, db - \frac{1}{2\pi}\iiint \frac{1}{r}\cdot\frac{d\zeta_a}{dc}\, da\, db\, dc.$$

Adding these three equations, and again calling the surface element of S $d\omega$, we obtain:

$$\frac{dL}{dx} + \frac{dM}{dy} + \frac{dN}{dz} = \frac{1}{2\pi}\int (\xi_a \cos\alpha + \eta_a \cos\beta + \zeta_a \cos\gamma)\,\frac{1}{r}\, d\omega$$
$$- \frac{1}{2\pi}\iiint \frac{1}{r}\left(\frac{d\xi_a}{da} + \frac{d\eta_a}{db} + \frac{d\zeta_a}{dc}\right) da\, db\, dc.$$

Since, however, throughout the interior of the space:

$$\frac{d\xi_a}{da} + \frac{d\eta_a}{db} + \frac{d\zeta_a}{dc} = 0, \qquad (2a)$$

and on its entire surface:

$$\xi_a \cos\alpha + \eta_a \cos\beta + \zeta_a \cos\gamma = 0, \qquad (2b)$$

therefore both integrals are equal to zero and equation (5b) as well as equations (2) are satisfied. Equations (4) and (5) or (5a) are thus indeed integrals of equations $(1)_4$ and (2).

The analogy mentioned in the introduction between the distance-actions of vortex filaments and the electromagnetic distance-actions of current-conducting wires, which provides a very good means of clearly demonstrating the form of vortex motions, is deducible from these theorems.

If we substitute in equation (4) the values of L, M, N from equations (5a), and denote by Δu, Δv, Δw those infinitely small parts of u, v, and w which in the integrals result from the element da, db, dc, also their resultant by Δp, we have:

$$\Delta u = \frac{1}{2\pi} \frac{(y-b)\zeta_a - (z-c)\eta_a}{r^3} \, da\, db\, dc,$$
$$\Delta v = \frac{1}{2\pi} \frac{(z-c)\xi_a - (x-a)\zeta_a}{r^3} \, da\, db\, dc,$$
$$\Delta w = \frac{1}{2\pi} \frac{(x-a)\eta_a - (y-b)\xi_a}{r^3} \, da\, db\, dc.$$

From these equations it follows that:

$$\Delta u (x-a) + \Delta v (y-b) + \Delta w (z-c) = 0,$$

that is, the resultant Δp of Δu, Δv and Δw is at right angles to r. Further:

$$\xi_a \Delta u + \eta_a \Delta v + \zeta_a \Delta w = 0,$$

that is, the same resultant Δp is also at right angles to the resulting axis of rotation at a, b, c. Finally:

$$\Delta p = \sqrt{\Delta u^2 + \Delta v^2 + \Delta w^2} = \frac{da\, db\, dc}{2\pi r^2} \sigma \sin \nu,$$

where σ is the resultant of ξ_a, η_a, ζ_a and ν the angle it makes with r, which is determined by means of the equation:

$$\sigma r \cos \nu = (x-a)\xi_a + (y-b)\eta_a + (z-c)\zeta_a .$$

Each rotating water particle a *thus determines in every other particle* b *of the same water mass a velocity whose direction is perpendicular to the plane through the axis of rotation of* a *and particle* b. *The magnitude of this velocity is directly proportional to the volume of* a, *its velocity of rotation, and the sine of the angle between the line* ab *and the axis of rotation, and inversely proportional to the square of the distance between both particles.*

Exactly the same law holds for the force that would be exerted by an electric current at a, parallel to the axis of rotation, on a magnetic particle at b.

The mathematical similarity of these two classes of natural phenomena rests upon this, that in the case of water vortices, for those parts of the water mass that have no rotation, a velocity potential exists that satisfies the equation:

$$\frac{d^2\varphi}{dx^2}+\frac{d^2\varphi}{dy^2}+\frac{d^2\varphi}{dz^2}=0$$

which equation holds everywhere except within the vortex filaments. If however, we consider the vortex filaments as always closed either within or outside of the water mass, then the space for which the differential equation for φ is valid is multiply connected, for it remains connected, if we imagine surfaces of separation through it, each of which is completely bounded by a vortex filament. In such multiply connected spaces a function φ that satisfies the above differential equation can become multivalued; and it must become multivalued if it is to represent currents returning into themselves; for, since the velocities of the water mass outside the vortex filaments are proportional to the differential quotients of φ, following the motion of the water one must progress to ever increasing values of φ. Therefore, if the current returns into itself, and if following it one finally arrives at the place where one had been previously, one finds for this place a second higher value of φ. Since the same procedure can be carried out infinitely often, there must exist infinitely many different values of φ for each point of such a multiply connected space that differ by the same differences, much as in the case of the different values of Arc tan $[x/y]$, which is such a multivalued function satisfying the above differential equation.

Such also is the case with the electromagnetic effects of a closed electric current. This acts at a distance just as a specific distribution of magnetic masses on a surface bounded by the conductor. Outside the current, therefore, the forces it exerts on a magnetic particle can be considered as the differential quotients of a potential function V that satisfies the equation:

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=0.$$

But in this case, too, the space that surrounds the closed conductor and in which this equation holds, is multiply connected, and V is multivalued.

Thus, in the case of vortex motions of water as in the case of electromagnetic effects, the velocities or forces outside the space traversed by vortex filaments or electric currents depend upon multivalued potential functions, which incidentally satisfy the general differential equation for magnetic potential functions, while inside the space penetrated by vortex filaments or electric currents instead of the potential functions, which do not exist here, different common functions of the kind appearing in equations (4), (5), and (5a) arise. On the other hand, in the case of simply streaming water motions and magnetic forces we are dealing with single-valued potential functions just as in the cases of gravitation, forces of electric attraction, and constant electric and thermal currents.

Those integrals of the hydrodynamical equations for which a single-valued velocity potential exists, we may call integrals of the first class. On the other hand, those where there is rotation of some of the water particles and consequently a multivalued velocity potential in the nonrotating water particles, we may call integrals of the second class. It may occur that in the latter case only those portions of the space are to be treated in the problem that contain no rotating water particles; for instance, in the case of motions of water in ring-shaped vessels, where a vortex filament may be supposed to lie along the axis of the vessel, and where the problem, therefore, still belongs to those that can be solved by the assumption of a velocity potential.

In the hydrodynamical integrals of the first class, the velocities of the water particles are in the same direction as and proportional to the forces that a specific distribution of magnetic masses outside the fluid would exert on a magnetic particle at the place of the water particle.

In the hydrodynamic integrals of the second class the velocities of the water particles are in the same direction as and proportional to the forces that would act on a magnetic particle and that would be produced by closed electric currents flowing through the vortex filaments with a density proportional to the velocity of rotation of these filaments, combined with magnetic masses outside the fluid. The electric currents inside the fluid would have to move with their respective vortex filaments and retain constant intensity. The assumed distribution of magnetic masses outside the fluid or on its surface must be taken so that the boundary conditions are satisfied. Each magnetic mass also, as we know, can be replaced by electric currents. Thus, instead of using for the values of u, v, w, the potential function P of an external mass k, we obtain a solution of the same generality if we give ξ, η, and ζ outside of, or even just at the surface of, the fluid arbitrary values such that only closed current filaments are generated; and then the integration in equations (5a) must be extended over the whole space in which ξ, η, and ζ are different from zero.

4. VORTEX SURFACES AND ENERGY OF VORTEX FILAMENTS

In hydrodynamic integrals of the first class, it is sufficient, as I have shown above, to know the motion of the surface. By this the motion in the interior of the fluid is entirely determined. In integrals of the second class, on the other hand, it is necessary, in addition, to determine the motion of the vortex filament in the interior of the fluid taking into account their mutual interaction and respecting the boundary conditions, which makes the problem much more complicated. Even this problem, however, can be solved for certain simple cases — specifically, when rotation of the water particles occurs only in certain surfaces or lines and the form of these surfaces or lines remains unchanged during the motion.

The properties of the surfaces bounded by an infinitely thin layer of rotating water particles can be deduced easily from equations (5a). If ξ, η, and ζ differ from zero only in an infinitely thin layer, their potential functions L, M, and N, according to known theorems, will have the same values on both sides of the layer, while their differential quotients, taken in the direction of the normal of the layer, will be different. If we assume the coordinate axes so placed that at the point of the vortex surface under consideration the z-axis corresponds to the normal of the surface and the x-axis to the axis of rotation of the water particles in the surface so that at this point $\eta = \zeta = 0$, then the potentials M and N as well as their differential quotients will have the same values on both sides of the layer. The same holds for L and dL/dx and dL/dy, while dL/dz will have two different values, whose difference is equal to $z\,\xi\,\varepsilon$, if ε denotes the thickness of the layer. Consequently equations (4), for u and w, yield the same values on both sides of the vortex surface, for v, however, values which differ by $2\xi\varepsilon$. Hence, that component of the velocity that is a tangent to the vortex surface and at right angles to the vortex lines differs in value on both sides of the surface. Within the layer of rotating water particles, the component of the velocity under consideration must be thought of as uniformly increasing from the value on one side of the surface to that on the other. For if ξ is constant here through the entire thickness of the layer, and α represents a proper fraction, v^1 the value of v on one, v_1 on the other side, v_α its value in the layer itself at a distance $\alpha\epsilon$ from the first side, then we saw that $v' - v_1 = 2\xi\epsilon$, because between both sides there is a layer of thickness ϵ and of intensity of rotation ξ. For the same reason we must have $v' - v_\alpha = 2\xi\epsilon\alpha = \alpha(v' - v_1)$, which expresses the above theorem. Since we must consider the rotating water particles as themselves moved, and the change of their distribution on the surface depends on their motion, we must assign to them a mean velocity of flow along the surface for the entire thickness of the layer, which corresponds to the arithmetical mean of the velocities on both sides of the layer.

Such a vortex surface would be produced, for example, when two previously separate moving masses of fluid come into contact with each other. At the surface of contact the velocities perpendicular to the surface necessarily would have to become equal. However, the velocities tangential to the surface will be different, in general, in the two fluid masses. The surface of contact thus would have the properties of a vortex surface.

On the other hand, isolated vortex filaments cannot, in general, be supposed infinitely thin, since the velocities at opposite sides of the filament then would attain infinitely great and opposite values, and the velocity of the filament itself would become indefinite. To nonetheless obtain certain general conclusions about the motion of very thin filaments of arbitrary cross section, we will make use of the principle of the conservation of vis viva.

Thus, before we proceed to specific examples, we will first form the equation for the vis viva K of the moving mass of water:

$$K = \tfrac{1}{2} h \iiint (u^2 + v^2 + w^2)\, dx\, dy\, dz.$$

We now from equations (4) substitute in this integral:

$$u^2 = u\left(\frac{dP}{dx} + \frac{dN}{dy} - \frac{dM}{dz}\right),$$

$$v^2 = v\left(\frac{dP}{dy} + \frac{dL}{dz} - \frac{dN}{dx}\right),$$

$$w^2 = w\left(\frac{dP}{dz} + \frac{dM}{dx} - \frac{dL}{dy}\right)$$

and integrate by parts, denoting by $\cos\alpha$, $\cos\beta$, $\cos\gamma$, and $\cos\vartheta$ the angles, which the inwardly directed normal of the element $d\omega$ of the water mass makes with the coordinate axes and with the resultant velocity q; we thus obtain, taking into account equations (2) and (1) :

$$K = -\frac{h}{2}\int d\omega\,[\,Pq\cos\vartheta + L(v\cos\gamma - w\cos\beta) \tag{6 a}$$
$$+ M(w\cos\alpha - u\cos\gamma) + N(u\cos\beta - v\cos\alpha)]$$
$$- h\iiint (L\xi + M\eta + N\zeta)\, dx\, dy\, dz.$$

The value of dK/dt is obtained from equations (1) by multiplying the first by u, the second by v, the third by w, and adding:

$$h\left(u\frac{du}{dt} + v\frac{dv}{dt} + w\frac{dw}{dt}\right) = -\left(u\frac{dp}{dx} + v\frac{dp}{dy} + w\frac{dp}{dz}\right)$$
$$+ h\left(u\frac{dV}{dx} + v\frac{dV}{dy} + w\frac{dV}{dz}\right) - \frac{h}{2}\left(u\frac{d(q^2)}{dx} + v\frac{d(q^2)}{dy} + w\frac{d(q^2)}{dz}\right).$$

if both sides are multiplied by $dxdydz$ and then we integrate over the entire extent of the water mass, noticing that because of $(1)_4$:

$$\iiint\left(u\frac{d\psi}{dx} + v\frac{d\psi}{dy} + w\frac{d\psi}{dz}\right)dx\,dy\,dz = -\int \psi\, q\cos\vartheta\, d\omega,$$

if ψ denotes a continuous and single valued function in the interior of the water mass, we obtain:

$$\frac{dK}{dt} = \int d\omega\,(p - hU + \tfrac{1}{2}hq^2)\, q\cos\vartheta. \tag{6 b}$$

If the water mass is entirely enclosed within rigid walls, $q\cos\vartheta$ must be zero at all points on the surface. Hence also $dK/dt = 0$, that is, $K = \text{constant}$.

If we consider this rigid wall as being at an infinite distance from the origin of the coordinates, and all existing vortex filaments at a finite distance, then the potential functions L, M, N, whose masses ξ, η, ζ are each in sum equal to zero, will at an infinite distance $\mathfrak{R}$ decrease proportional to $\mathfrak{R}^{-2}$, and the velocities, their differential quotients, will decrease as $\mathfrak{R}^{-3}$,; but the surface element dw if it is always to correspond to the same solid angle at the coordinate origin, will increase proportional to $\mathfrak{R}^2$. The first integral in the expression for K[equation (6a)], which is extended over the surface of the water mass, will decrease as $\mathfrak{R}^{-3}$, and, therefore, will vanish for an infinite value of $\mathfrak{R}$. Thus the value of K reduces to:

$$K = -h \iiint (L\xi + M\eta + N\zeta)\,dx\,dy\,dz \qquad (6c)$$

and this value does not alter during the motion.

5. STRAIGHT PARALLEL VORTEX FILAMENTS

First we shall consider the case where only straight vortex filaments parallel to the z-axis exist, whether in an infinitely extended mass of water or in a similar mass limited by two infinite planes perpendicular to the filaments, which amounts to the same thing. All motions then occur in planes perpendicular to the z-axis and are exactly the same in all such planes.

We therefore put:

$$w = \frac{du}{dz} = \frac{dv}{dz} = \frac{dp}{dz} = \frac{dV}{dz} = 0.$$

Then equations (2) reduce to:

$$\xi = 0, \quad \eta = 0, \quad 2\zeta = \frac{du}{dy} - \frac{dv}{dx},$$

equations (3) to:

$$\frac{\delta\zeta}{\partial t} = 0.$$

The vortex filaments thus retain constant rotational velocity as well as constant cross section.

Equations (4) reduce to:

$$u = \frac{dN}{dy}, \qquad v = -\frac{dN}{dx}, \quad \frac{d^2N}{dx^2} + \frac{d^2N}{dy^2} = 2\zeta.$$

In accordance with the remark at the end of section 3, we have here put $P=0$. The equation of the stream lines is therefore N=constant.

N is in this case the potential function of infinitely long lines; it is itself infinitely great, but its differential quotients are finite. If a and b are the coordinates of a vortex filament, whose cross section is *da db*, then we have:

$$-v = \frac{dN}{dx} = \frac{\zeta\, da\, db}{\pi} \cdot \frac{x-a}{r^2}, \quad u = \frac{dN}{dy} = \frac{\zeta\, da\, db}{\pi} \cdot \frac{y-b}{r^2}.$$

From this it follows that the resultant velocity q is perpendicular to r, which is the perpendicular to the vortex filament, and that:

$$q = \frac{\zeta\, da\, db}{\pi r}.$$

If in a water mass infinitely extended in the directions of x and y we have several vortex filaments, whose coordinates are x ,y ,x ,y , and so forth, and we denote the product of the rotational velocity and the cross section of each of these by m ,m and so forth, then, forming the sums:

$$U = m_1 u_1 + m_2 u_2 + m_3 u_3 \text{ etc.},$$
$$V = m_1 v_1 + m_2 v_2 + m_3 v_3 \text{ etc.},$$

these will each be equal to zero, since the portion of the sum V that arises from the effect of the second vortex filament on the first is canceled by the effect of the first on the second. For both are:

$$m_1 \cdot \frac{m_2}{\pi} \frac{x_1 - x_2}{r^2} \text{ und } m_2 \cdot \frac{m_1}{\pi} \frac{x_2 - x_1}{r^2}$$

and so for all the others in both sums. Now U is the velocity of the center of gravity of the masses $\mathbf{m}_1$, $\mathbf{m}_2$ and so forth, in the direction of $\mathbf{x}$ multiplied by the sum of these masses; so of V in the direction of y. Both velocities are thus zero, unless the sum of the masses is zero, in which case there is no center of gravity. Thus the center of gravity of the vortex filaments remains unchanged during their motions about one another; and since this theorem holds for any arbitrary distribution of vortex filaments we also may apply it to isolated vortex filaments of infinitely small cross section.

From this we derive the following consequences:

1. In case of a single rectilinear vortex filament of infinitely small cross section in a water mass infinite in all directions perpendicular to the vortex filament, the motion of the water particles at a finite distance from it depends only on the product $\zeta\, da\, db = m$ of the rotational velocity and the magnitude of its cross section, not on the form of its cross section. The particles of the water mass rotate about it with tangential velocity $m/\pi r$, where r denotes the distance from the center of gravity of the vortex filament. The position of the center of gravity itself, the rotational velocity, the magnitude of the cross section, and thus also the magnitude m remain unchanged, even if the form of the infinitely small cross section may alter.

2. In case of two rectilinear vortex filaments of infinitely small cross section in an unlimited water mass, each will cause the other to move in a direction perpendicular to the line connecting them. The length of the connecting line is not changed as a result of this. Thus, both will rotate about their common center of gravity at constant distances from it. If the rotational velocity in both vortex filaments is of the same direction, that is, of the same sign, then their center of gravity must lie between them. If it is of opposite direction, that is, of different signs, then their center of gravity lies in the prolongation of the line connecting them. And if the product of the rotational velocity and the cross sections is the same for both, but of opposite sign, so that the center of gravity would lie at an infinite distance, they both travel forward with equal velocity and in the same direction perpendicular to the line connecting them.

To the latter case may also be referred that in which a vortex filament of infinitely small cross section moves next to an infinitely extended plane that is parallel to it. The boundary condition for the motion of the water touching the plane, that is, that it must be parallel to the plane, is satisfied by imagining that beyond the plane there is a second vortex filament, the mirror image of the first. From this it follows that the vortex filament in the water mass travels forward parallel to the plane in the direction in which the water particles between it and plane move, and with one-fourth of the velocity possessed by the water particles at the foot of a perpendicular from the vortex filament onto the plane.

For rectilinear vortex filaments the assumption of an infinitely small cross section leads to no inadmissible consequences, since no individual filament exerts a propelling action upon itself, but is propelled only by the influence of the other filaments present. It is different for curved filaments.

6. CIRCULAR VORTEX FILAMENTS

Assume that in an infinitely extended water mass there exist only two circular vortex filaments, whose planes are perpendicular to the z-axis and whose centers lie in this axis so that all around them everything is symmetrical. Let the coordinates be changed by putting:

$$\begin{aligned} x &= \chi \cos \varepsilon, & a &= g \cos e, \\ y &= \chi \sin \varepsilon, & b &= g \sin e, \\ z &= z, & c &= c. \end{aligned}$$

According to our assumption the velocity of rotation σ is a function only of χ and z or of g and c, and the axis of rotation is everywhere perpendicular to χ (or g) and the z-axis. Thus the rectangular components of the rotation at the point with coordinates g, e, and c are:

$$\xi = -\sigma \sin e, \quad \eta = \sigma \cos e, \quad \zeta = 0.$$

In equations (5a) we obtain:

$$r^2 = (z-c)^2 + \chi^2 + g^2 - 2\chi g \cos(\varepsilon - e),$$
$$L = \frac{1}{2\pi}\iiint \frac{\sigma \sin e}{r} g\, dg\, de\, dc,$$
$$M = -\frac{1}{2\pi}\iiint \frac{\sigma \cos e}{r} g\, dg\, de\, dc,$$
$$N = 0.$$

By multiplying with $\cos\varepsilon$ and $\sin\varepsilon$, and adding, one obtains from the equations for L and M:

$$L\sin\varepsilon - M\cos\varepsilon = -\frac{1}{2\pi}\iiint \frac{\sigma\cos(\varepsilon-e)}{r} g\, dg\, d(\varepsilon-e)\, dc,$$
$$L\cos\varepsilon + M\sin\varepsilon = \frac{1}{2\pi}\iiint \frac{\sigma\sin(\varepsilon-e)}{r} g\, dg\, d(\varepsilon-e)\, dc.$$

In both integrals the angles e and ε occur only in the form of ($\varepsilon - e$), and this magnitude, therefore can be regarded as the variable under the integral. In the second integral, the elements in which $(\varepsilon - e) = \mathfrak{e}$ are canceled by those in which $(\varepsilon - e) = 2\pi - \mathfrak{e}$, and it is, therefore, equal to zero. If we put:

$$\psi = \frac{1}{2\pi}\iiint \frac{\sigma\cos e \,.\, g\, dg\, de\, dc}{\sqrt{(z-c)^2 + \chi^2 + g^2 - 2g\chi\cos e}}, \tag{7}$$

we therefore obtain:

$$M\cos\varepsilon - L\sin\varepsilon = \psi,$$
$$M\sin\varepsilon + L\cos\varepsilon = 0,$$

or:

$$L = -\psi\sin\varepsilon, \qquad M = \psi\cos\varepsilon. \tag{7a}$$

Calling τ the velocity in the direction of the radius χ, and taking into account that because of the symmetrical position of the vortex rings about the axis the velocity in the direction of the circumference must be equal to zero, we have:

$$u = \tau\cos\varepsilon, \qquad v = \tau\sin\varepsilon,$$

and according to equations (4):

$$u = \frac{dM}{dz}, \qquad v = \frac{dL}{dz}, \qquad w = \frac{dM}{dx} - \frac{dL}{dy}.$$

From this it follows:

$$\tau = -\frac{d\psi}{dz}, \qquad w = \frac{d\psi}{d\chi} + \frac{\psi}{\chi},$$

or

$$\tau\chi = -\frac{d(\psi\chi)}{dz}, \qquad w\chi = \frac{d(\psi\chi)}{d\chi}. \tag{7b}$$

The equation of the streamlines is therefore:

$$\psi\chi = \text{Const.}$$

If first we carry out the integration indicated in the value of ψ for a

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or

$$\tau\chi = -\frac{d(\psi\chi)}{dz}, \qquad w\chi = \frac{d(\psi\chi)}{d\chi}. \quad (7b)$$

The equation of the streamlines is therefore:

$$\psi\chi = \text{Const.}$$

If first we carry out the integration indicated in the value of ψ for a

vortex filament of infinitely small cross section, putting and labeling the part of ψ obtained from this by ψ_{m_1}, then: $\sigma dg\, dc = m_1$,

$$\psi_{m_1} = \frac{m_1}{\pi}\sqrt{\frac{g}{\chi}}\left\{\frac{2}{\varkappa}(F-E) - \varkappa F\right\},$$

$$\varkappa^2 = \frac{4g\chi}{(g+\chi)^2 + (z-c)^2},$$

where F and E are the complete elliptic integrals of the first and second kind for the modulus

Putting, for the sake of brevity,

$$U = \frac{2}{\varkappa}(F-E) - \varkappa F,$$

where, therefore, U is a function of $\varkappa$, we have:

$$\tau\chi = \frac{m_1}{\pi}\sqrt{g\chi}\,\frac{dU}{d\varkappa}\cdot\varkappa\cdot\frac{z-c}{(g+\chi)^2+(z-c)^2}\,.$$

Now, if at the point given by χ and z there exists a second vortex filament m, and if by τ_1 we denote the velocity in the direction of g that it imparts to the vortex filament m_1, then we obtain the latter if in the expression for τ we put $\tau_1\ g\ \chi\ c\ z\ m.$ instead of $\tau\ \chi\ g\ z\ c\ m_1$

In this $\varkappa$ and U remain unchanged and we obtain:

$$m\tau\chi + m_1\tau_1 g = 0. \tag{8}$$

If now we determine the value w of the velocity parallel to the axis, which is produced by the vortex filament m_1, whose coordinates are g and c, we find:

$$w\chi = \frac{1}{2}\,\frac{m_1}{\pi}\sqrt{\frac{g}{\chi}}\,U + \frac{m_1}{\pi}\sqrt{g\chi}\,\frac{dU}{d\varkappa}\cdot\frac{\varkappa}{2\chi}\cdot\frac{(z-c)^2+g^2-\chi^2}{(g+\chi)^2+(z-c)^2}\,.$$

If we call w_1 the velocity parallel to the z-axis produced by the vortex ring m, whose coordinates are z and χ, at the position of m_1, then we need only make again the previously indicated interchange of the relevant coordinates and masses. Then one finds that:

$$2mw\chi^2 + 2m_1w_1g^2 - m\tau\chi z - m_1\tau_1 gc = \frac{2mm_1}{\pi}\sqrt{g\chi}\,U. \tag{8a}$$

Sums similar to (8) and (8a) can be formed for an arbitrarily large number of vortex rings. For the nth of these the product $\sigma\, dg\, dc$ is denoted by m_n, the components of the velocity, that is imparted to it by the other vortex filaments by τ_n and w_n, where, however, we omit for the time being the velocities that each vortex ring can impart to itself. I also call the radius of the ring ϱ_n and λ, its distance from a plane perpendicular to the axis, which two magnitudes coincide in direction with χ and z, but as belonging to a particular vortex ring are functions of the time and not independent variables like χ and z. Finally, let the value of ψ, so far as it arises from the other vortex rings, be ψ_n. From (8) and (8a), by writing out the corresponding equations for every single pair of vortex rings and adding all of them, we obtain:

$$\Sigma[m_n\varrho_n\tau_n] = 0,$$
$$\Sigma[2m_nw_n\varrho_n^2 - m_n\tau_n\varrho_n\lambda_n] = \Sigma[m_n\varrho_n\psi_n].$$

As long as in these sums we still have a finite number of separate and infinitely thin vortex rings, we can consider w, τ, and ψ only as those parts of these magnitudes that are produced by the presence of the other rings. If, however, we suppose the space to be continuously filled with an infinitely large number of such rings, ψ is the potential function of a continuous mass, w and τ are the differential quotients of this potential function; and it is known that for such a function as well as for its differential quotients the parts of the function due to the presence of mass in an infinitely small neighborhood around the point for which the function is determined are infinitely small in comparison with those due to finite masses at a finite distance.*

If, therefore, we change the sums into integrals, then we can consider w, τ, and ψ the total values of these magnitudes at the point in question, and put:

$$w = \frac{d\lambda}{dt}, \quad \tau = \frac{d\varrho}{dt}$$

For this purpose we replace the magnitude m with the product $\sigma\, d\varrho\, d\lambda$.

$$\iint \sigma\varrho \frac{d\varrho}{dt} d\varrho\, d\lambda = 0, \tag{9}$$

$$2\iint \sigma\varrho^2 \frac{d\lambda}{dt} d\varrho\, d\lambda - \iint \sigma\varrho\lambda \frac{d\varrho}{dt} d\varrho\, d\lambda = \iint \sigma\varrho\psi\, d\varrho\, d\lambda. \tag{9a}$$

Since according to section 2 the product $\sigma\, d\varrho\, d\lambda$ is constant with respect to time, equation (9) can be integrated with respect to t and we obtain:

$$\tfrac{1}{2}\iint \sigma\varrho^2\, d\varrho\, d\lambda = \text{Const.}$$

If we consider the space to be divided by a plane that passes through the z-axis and therefore cuts all existing vortex rings; if we further consider σ as the density of a layer of mass, and call $\mathfrak{M}$ the entire mass in this layer of the plane, that is:

$$\mathfrak{M} = \iint \sigma\, d\varrho\, d\lambda,$$

and R^2 the mean value of ϱ^2 for all the elements of mass, then:

$$\iint \sigma\varrho \cdot \varrho\, d\varrho\, d\lambda = \mathfrak{M} R^2,$$

and since this integral and the value of $\mathfrak{M}$ are constant with respect to time, it follows that R, too, also remains unchanged during the forward motion.

Therefore, if there exists in the unlimited fluid mass only one circular vortex filament of infinitely small cross section, then its radius remains unchanged.

*Cf Gauss in *Resultate des magnetischen Vereins*, 1839, p. 7.

According to equation (6c) the magnitude of the vis viva is in our case:

$$\begin{aligned} K &= -h \iiint (L\xi + M\eta)\, da\, db\, dc \\ &= -h \iiint \psi\sigma \,.\, \varrho\, d\varrho\, d\lambda\, d\varepsilon \\ &= -2\pi h \iint \psi\, \sigma \,.\, \varrho\, d\varrho\, d\lambda. \end{aligned}$$

It, too, is constant with respect to time.

We further note that since $\sigma\, d\varrho\, d\lambda$. is constant with respect to time,

$$\frac{d}{dt}\iint \sigma\varrho^2\lambda\, d\varrho\, d\lambda = 2\iint \sigma\varrho\lambda\, \frac{d\varrho}{dt}\, d\varrho\, d\lambda + \iint \sigma\varrho^2 \frac{d\lambda}{dt}\, d\lambda\, d\varrho,$$

and thus equation (9a), if by l we denote the value of λ for the center of gravity of the cross section of the vortex filament and multiply (9) by it and add, becomes:

$$2\,\frac{d}{dt}\iint \sigma\varrho^2\lambda\, d\varrho\, d\lambda + 5\iint \sigma\varrho\,(l-\lambda)\,\frac{d\varrho}{dt}\, d\varrho\, d\lambda = -\frac{K}{2\pi h}. \quad (9b)$$

If the cross section of the vortex filament is infinitely small, and ε is an infinitely small magnitude of the same order as $l-\lambda$ and the other linear dimensions of the cross section, while $\sigma\, d\varrho\, d\lambda$ is finite, then ψ as well as K are of the same order of infinitely large quantities as $\log\, \varepsilon$. For very small values of the distances v from the vortex ring we have:

$$\begin{aligned} v &= \sqrt{(g-\chi)^2 + (z-c)^2}, \\ \varkappa^2 &= 1 - \frac{v^2}{4g^2}, \\ \psi_{m_1} &= \frac{m_1}{\pi}\log\left(\frac{\sqrt{1-\varkappa^2}}{4}\right) = \frac{m_1}{\pi}\log\frac{v}{8g}. \end{aligned}$$

In the value of K, ψ is further multiplied by ϱ or g. If g is finite and v of the same order as ϵ then K is of the order of $\log\, \varepsilon$. Only if g is infinitely large of the order of $1/\varepsilon$, K becomes infinitely large as $(1/\varepsilon)\log\, \varepsilon$. The the circle becomes a straight line. But, on the other hand, $d\varrho\,/\,dt$ which is equal to $d\psi/dz$, becomes of the order $1/\varepsilon$, the second integral therefore is finite and for finite ϱ vanishingly small compared to K. In this case we may put the constant l in place of λ, and obtain:

$$2\,\frac{d(\mathfrak{M}R^2 l)}{dt} = -\frac{K}{2\pi h}$$

or:

$$2\,\mathfrak{M}R^2 l = C - \frac{K}{2\pi h}\, t.$$

Since $\mathfrak{M}$ and R are constant, l can vary only proportional to time. If $\mathfrak{M}$ is positive, the motion of the water particles on the outer side of the ring is in the direction of positive z, on the inner side in the direction of negative z; K, h and R from their nature always positive.

From this, therefore, it follows that in case of a circular vortex filament of very small cross section in an infinitely extended water mass the center of gravity of the cross section has motion parallel to the axis of the vortex ring of approximately constant and very high velocity, which is directed to the same side to which the water flows through the ring. Infinitely thin vortex filaments of finite radius would attain infinitely great translational velocities. If, however, the radius of the vortex filament is infinitely great of order $1/\varepsilon$, then R^2 becomes infinitely great in comparison with K, and l becomes constant. The vortex filament, which now has changed into a straight line, becomes stationary, as we have already found earlier for rectilinear vortex filaments. Now we can also see in general how two ringlike vortex filaments with the same axis will mutually affect each other, since each in addition to its own motion also follows the motion of the water particles produced by the other. If they have the same direction of rotation, they both travel in the same direction; the one in front will widen and then travel more slowly, the pursuing one will contract and travel faster, till finally at velocities not too different, it will catch up with the other and go through it. Then this same game will be repeated with the other one so that, in turn, the rings will pass one through the other.

If the vortex filaments have equal radii and equal and opposite velocities of rotation, they will approach each other and widen one another, so that finally when they have come very close to each other their velocity of approach becomes smaller and smaller; the widening, on the other hand, occurs with increasing velocity. If the two vortex filaments are entirely symmetrical, the velocity of the water particles midway between the two and parallel to the axis is equal to zero. Thus one might imagine a rigid wall inserted here without disturbing the motion, and so obtain the case of a vortex ring that runs up against a rigid wall.

Finally, I remark that these motions of circular vortex rings are easily studied in nature by rapidly drawing for a short space along the surface of a fluid a half-immersed circular disc or the approximately semicircular point of a spoon and quickly withdrawing it. Half vortex rings then remain in the fluid whose axis lies in the free surface. The free surface thus forms a limiting plane of the water mass placed through the axis, but as a result of which there is no essential change in the motions. The vortex rings travel on, widen when they come to a wall, and are widened or contracted by other vortex rings exactly as we have deduced it from the theory.

RESEARCH REVIEW

Some New Directions in Fluid Mechanics

STEVEN BARDWELL

The history of fluid dynamics prior to 1972 divides itself neatly into two periods, with World War I marking the division. The so-called classical period of fluid dynamics from the beginning drew its empirical inspiration from the intuitively most striking feature of turbulence — the formation of

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EDITOR'S NOTE

The IJFE was founded with the aim of providing plasma physicists with cross-fertilization of conceptual work in areas of science that otherwise might go unnoticed. We are initiating this Research Review column with that goal in mind. In coming issues, we intend to review books in areas of mathematical physics with potentially important applications to plasma physics — but largely unknown to plasma physicists.

We inaugurate the column in this issue with a review of three recent articles in fluid mechanics, all dealing with the subject of coherent structures in fluids, from both an experimental and theoretical point of view. The relevance of this work to plasma physics is increasingly clear; the article by Dr. Wells in this issue concerns itself with similar lines of research in plasmas.

Norman J. Zabusky, "Coherent Structures in Fluid Dynamics," to be published in *Proceedings of Orbis Scientiae Conference* (Coral Gables, Fla.:University of Miami, 1977)

John Laufer, "New Trends in Experimental Turbulence Research," *Annual Review of Fluid Mechanics,* edited by M. van Dyke and W.G. Vincenti (Palo Alto, Cal.: Annual Reviews 1975)

George Lamb, Jr., "Solitons and the Motion of Helical Curves," *Physical Review Letters*, Vol. 19:235, 1976

vortices, eddies, and other large-scale, coherent motions. Aristotle, the notebooks of DaVinci, and the speculations of Descartes and Kant all show the profound impact of vortex motion on attempts to generalize our knowledge of continuum systems. The hydrodynamics of the late 19th and early 20th century concentrated on explaining the motion and stability of these large-scale structures.*

By the end of the 19th century, the basic equations of fluid dynamics had been exhaustively studied by the best mathematicians but were satisfactorily solved in only a very few cases. A.B. Basset, in his classic *Treatise on Hydrodynamics* (published in 1888 and quoted by Zabusky) summarized the problem that continued to befuddle these mathematicians: "The mathematical difficulties of solving this problem [the fluid equations] when the initial distribution of the vortices and the initial forms of their cross sections are given are very great; and it seems impossible in the present state of analysis to do more than obtain approximate solutions in certain cases."

Faced with these formidable difficulties, experimental and theoretical work gradually changed direction in the next 30 years. Based on the successes of statistical mechanics in dealing with systems that had a large number of degrees of freedom, fluid mechanics turned to statistical methods. The work of G.B. Taylor on theoretical and experimental studies of the statistical properties (especially spectra and correlation functions) in the 1920s and 1930s set the stage for what became known as the modern theory of turbulence. In the last 20 years, a number of mathematically sophisticated applications of various statistical many-body approximations have been made to fluid turbulence, primarily under the influence of Robert Kraichnan, and these theories have dominated the field of fluid mechanics.**

All these statistical theories are based on the observation that turbulence is a disordered flow of fluid, whose only predictable features are various means, moments, and correlation functions. The pervasive lesson of Boltzmann's kinetic theory of gases dominated the thinking in fluid dynamics. The greater the number of degrees of freedom, the greater the chance of disordered motion in the small and the greater the chance of statistically smooth behavior in the large. Both experimental and theoretical work was (and still is, to a large extent) motivated by the conviction that the classical quest for causality and predictability in complex systems had to be sacrificed for statistics. As Orzag put it:

> [A second] important characteristic of turbulent flows is their apparent randomness and instability to small perturbations. Two turbulent flows that are at some time nearly identical in detail do not remain nearly identical on the time scales of dynamical in-

*For an extensive bibliography on these vortex structures see Lamb's *Hydrodynamics*.

**For a comprehensive account of the modern statistical theory of turbulence, see Orszag's *Lecture Notes on the Statistical Theory of Turbulence*.

> terest. This property of turbulent flows may be used to give a quantitative definition of turbulence. Also, instability of turbulent motion is related to the limited "predictability"of atmospheric motions.
>
> While the details of fully developed turbulent motions are extremely sensitive to triggering disturbances, average properties are not. Otherwise there would be little significance in the averages. On the other hand, transition flows (which occur naturally at Reynolds numbers several times critical) have statistical properties which are sensitive to the nature of the perturbations. The idea that fully developed turbulent flows are extremely sensitive to small perturbations but have statistical properties that are insensitive to perturbations is of central importance....

There was just one persistent problem with these statistical theories: The quantities they could predict tended to be the least interesting, and, in general, they were unable to explain turbulent phenomena at all.* However, this fact is recognized only by a relatively small number of fluid dynamicists. The three papers discussed here, by Zabusky, Laufer, and Lamb, are the opening shots in a long-overdue reassessment of statistical approaches to physics. As Zabusky says:

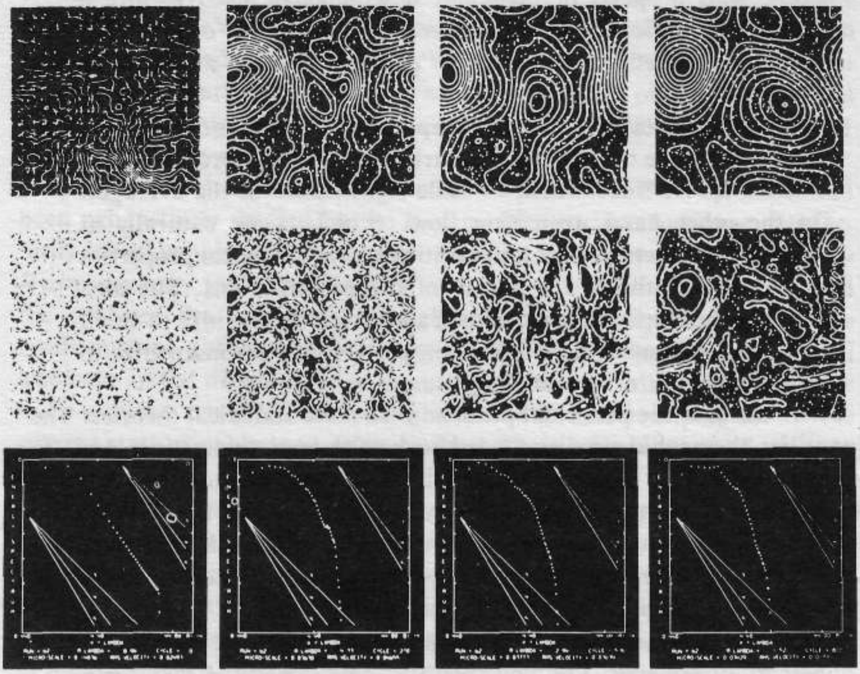
> In the last decade we have experienced a conceptual shift in our view of turbulence. For flows with strong velocity shears, near boundaries, density gradients, magnetic fields or other organizing characteristics, many now feel that the spectral or wave-number space description has inhibited fundamental progress.
>
> The next "El Dorado" lies in the mathematical understanding of coherent structures in weakly dissipative fluids: the formation, evolution, and interaction of metastable, vortex-like solutions of nonlinear, partial differential equations.

Laufer makes the same point from the point of view of an experimentalist in his paper:

> In the past 10 years two important observations were reported that had a significant impact on subsequent turbulence research (Kline and Runstadler 1959, Brown and Roshko 1971). Ironically, these were made not by sophisticated electronics instrumentation but visually with rather simple optical techniques. The essence of these observations was the discovery that turbulent flows of simple geometry are not so chaotic as had been previously assumed: There is some order in the motion with an observable chain of events recurring randomly with a statistically definable mean period. This surprising result encouraged researchers to reexamine the line of inquiry for designing their experiments, and they began seriously questioning the relevance of some of the statistical quantities they

*Saffman has an early critique of these theories.

FIGURE 1
COMPUTER-GENERATED SOLUTIONS TO THE FLUID EQUATIONS IN TWO DIMENSIONS FROM TAPPERT (1971).



Times moves from left to right. The upper frames are stream lines of the fluid. The second row is of the vorticity, a more sensitive measure than the stream function of the small-scale motions of the fluid. The third row shows the energy spectrum as a function of inverse length. Qualitatively the general tendency for small vortices to clump together into large ones is evident. The spectrum shows this quantitatively, as the predominant motion is for energy to shift from right to left (from small length scales to large).

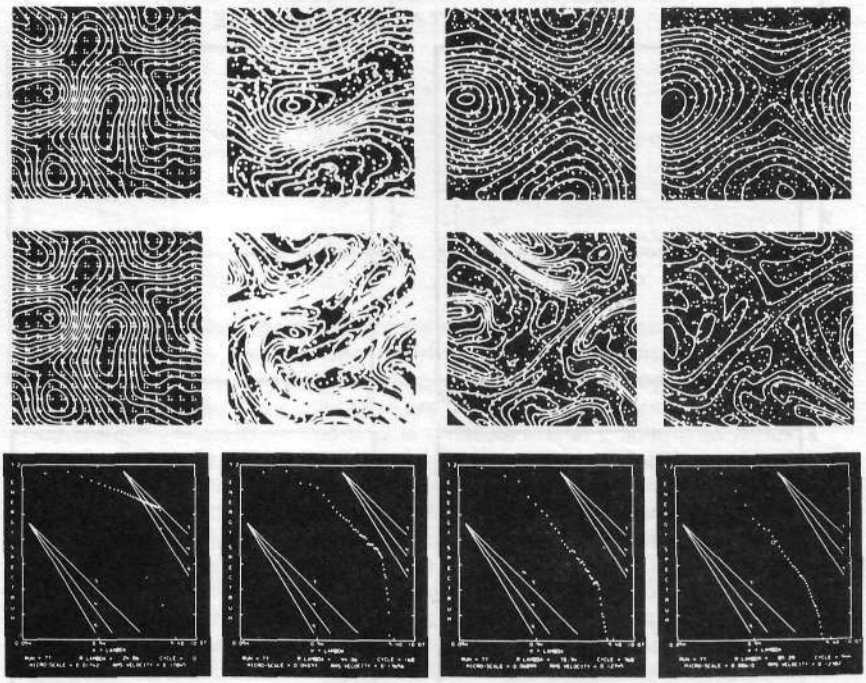
had been measuring. It was soon realized, for instance, that retaining some phase information in the statistics and obtaining more detailed spatial information are essential for a quantitative explanation of the visual observations.

The discontent with statistical, nonpredictive approaches to fluid dynamics has come to a head in the past several years, with the coalescence of empirical data (especially in superfluids, boundary layer mixing experiments, and measurements of wakes), numerical simulations, * and theoretical work.** In all these areas it became obvious that statistical averaging was in fact destroying the most interesting and important phenomena in turbulence — the formation, dynamics, and persistence of vortex motion.

*Zabusky describes the "supra-nova-like experience which revealed key links and exposed new analytical and computational directions" upon seeing the set of computer-generated films on the formation of vortices. (See Figure 1)

**Hasimoto's 1972 paper on the soliton on a vortex filament leads directly to Lamb's paper.

COMPUTER-GENERATED SOLUTIONS TO THE MAGNETO-HYDRODYNAMIC EQUATIONS IN TWO DIMENSIONS FROM TAPPERT (1971).



As above, time moves from left to right, and the top row is stream lines. Here, however, in the plasma the second row shows the magnetic field lines. The third, again, shows the energy spectrum. Notice, in addition to the formation of large vortical structures, the way that the stream lines and magnetic lines orient parallel to each other to form a force-free structure.

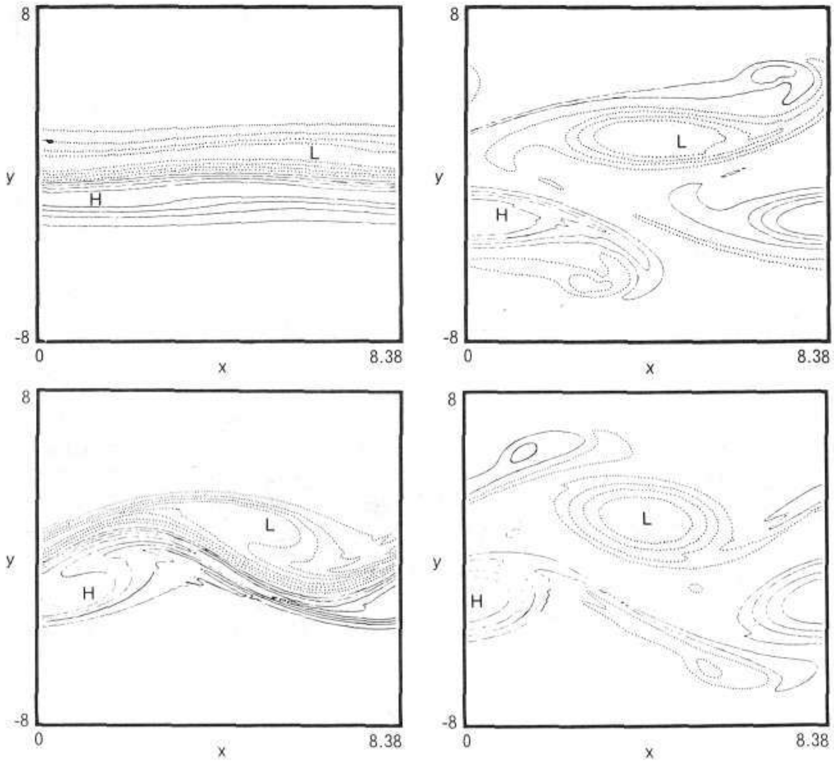
The challenge taken up by the three papers reviewed here is that of resurrecting the classical approach to fluid dynamics, but in a form and with new tools that can surmount the mathematical difficulties described by Basset.

NEW DIRECTIONS IN FLUID DYNAMICS

Zabusky's paper takes up the challenge of describing large-scale, coherent structure in fluids, with the hope that a synergistic interaction of high-speed numerical solutions of appropriately posed problems can provide insights into the relevant features of vortex motion and that these insights can then be extracted analytically. In specifically analyzing the problem of finite-area vortex motion (in contrast with point — or, more properly, line — vortices), Zabusky presents an impressive case that this hybrid technique can accomplish much fruitful work. There has been progress in five important areas, he says, all previously unapproachable: the generation of finite-area vortices, the growth of perturbations on the surface of these vortices, the slow pitching motion of these vortices, the breakdown of systems of finite-area vortices, and the persistence of these vortices at high Reynolds numbers.

FIGURE 2

CONSTANT VORTICITY PROFILES (NINE LEVELS)
FOR A SLIGHTLY PERTURBED GAUSSIAN PROFILE,
OF VELOCITY PAST A LONG THIN PLATE

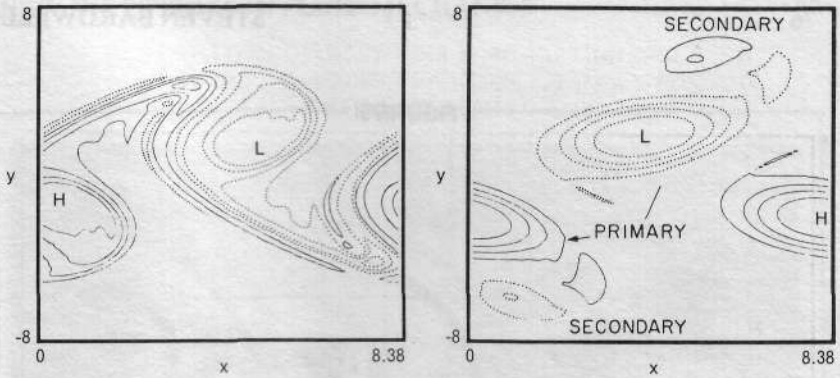


Zabusky details a number of phenomena *first* uncovered and then described by the combination of numerical and theoretical analysis; for example, the nutation, or slow rocking motion of the vortices that form in the wake of a flat plate. A number of experiments had been done that measured vortex formation in the wake, but until some computer calculations by Zabusky and others several years after the experiments, this periodic oscillation went unnoticed. Just recently an analytical calculation has derived the nutation frequency from the Navier-Stokes equations (see Figure 2).

All these problems are untreatable by conventional methods, primarily because of the significant increase in degrees of freedom allowed by surface deformations. Contrary to conventional wisdom, the increasing number of degrees of freedom tends to stabilize the system of vortices! This multifaceted result in hydrodynamics is perhaps the most important feature of these recent studies. The statistical intuition that a system with a large number of degrees of freedom should be entropic is fundamentally inapplicable.

Zabusky quotes von Neumann who said with considerable prescience in 1946:

> Our present analytical methods seem unsuitable for the important problems arising in connection with nonlinear partial



Zabusky describes the results, modeled after an experiment by Sato and Kuriki (1961): "In frame a one clearly sees that the derivative of the Gaussian changes sign, that is the upper dotted lines are negative regions of vorticity and the lower solid lines are positive. Also apparent is the long wavelength perturbation. The flow is unstable and at c finite area vortex regions (FAVR) have formed, joined by elongated and structured appendages or 'arms.' The Reynolds number is 750 and weak but finite dissipation smooths the flow, as is evident in subsequent frames. Two new features emerge in d and persist. First, regions of opposite signed vorticity have been *entrained* by the dominant or 'primary' FAVRs. That is, in the upper-half of the frames one sees secondary regions of vorticity moving past primary FAVRs (solid contours past dotted contours). The second feature is the seemingly *elliptical* vortex region (exaggerated because of different vertical-horizontal figure scales) whose major axis undergoes a pitching or 'nutation' with respect to the horizontal x axis."

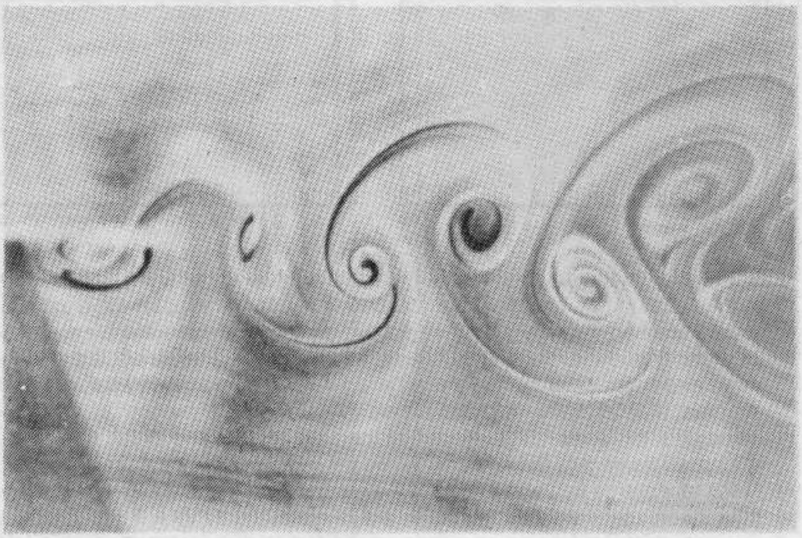
*Source:*Zabusky

> differential equations and in fact with virtually all types of nonlinear problems.... We are up against an important conceptual difficulty which tends to obscure the great physical and mathematical regularities that do exist....
>
> ...Really efficient high-speed computing devices may, in the field of nonlinear partial differential equations as well as in many other fields which are now difficult or entirely denied of access, provide us with those heuristic hints which are needed in all parts of mathematics for genuine progress ... and should ultimately lead to important analytical advances.

Laufer pursues the same line of thinking in his review of new directions in experimental work in fluid dynamics. The critical point Laufer makes is that the change in philosophy has resulted in the use of a new set of experimental techniques (along with more sophisticated data reduction). Since the statistical measurements that had occupied experimental work in the past proved unable to account for the observed large-scale regularities, the same problems noted above by Zabusky were studied using new, mainly optical, techniques such as high-speed photography, Schlieren photography, and shadowgraphs.

The accompanying photography of the wake of an oscillating cylinder is one of many studies that have dealt with the formation, dynamics, and

FIGURE 3



Photographs of the wake behind a vibrating cylinder from Griffin and Ramberg 1974. The Reynolds number is 190, and the frequency of the cylinder 36.6 hertz.

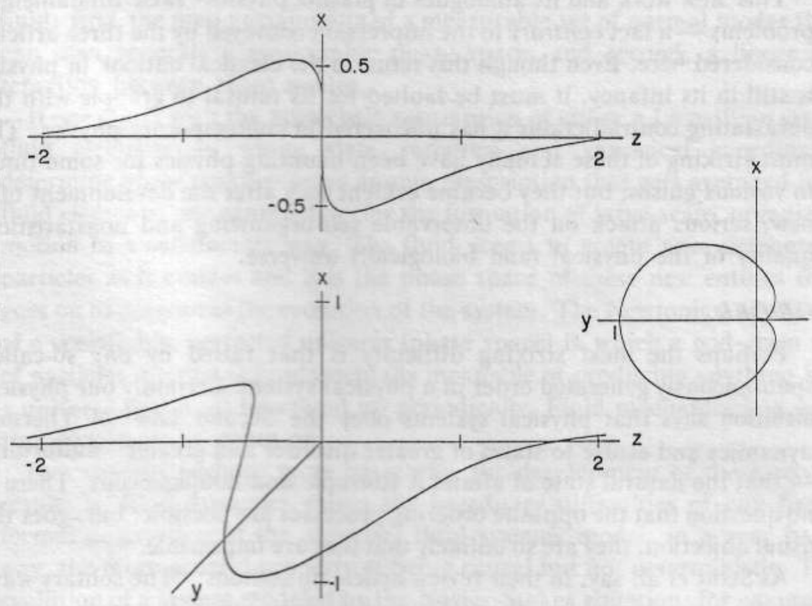
Source: Journal of Fluid Mechanics

decay of vortex systems (see Figure 3). In such seemingly simple phenomena, which have been the object of very detailed statistical studies in the past as boundary layer mixing (and were thought to be well understood!), the discovery of large-scale vortex motion in the boundary layer came as a great surprise. The study of what Laufer calls quasi-ordered structures is absolutely essential to understand the behavior of what had been mistakenly thought of as random phenomena.

Lamb's work represents a third, distinct direction in response to the same recognition that organized structure in fluids is not an anomaly but the basic reality of all fluid motion. His work especially has an immediate bearing on plasma physics because he develops a profound, theoretical connection between two general classes of structure that occur in plasmas: vortex motion (circulation cells, and so on) and solitons.

In an unmagnetized plasma, the physical mechanisms behind these phenomena are entirely different. The magnetohydrodynamic structures like vortices come out of the motion of the heavy component of the plasma, the ions, and seem to depend on the electrons in an entirely unimportant way. Such vortices have characteristically long time and spatial scales. In the case of solitons, the important dynamics come from the rapid motion of the electrons and the ions react on a much slower time scale determined by the electron motion. In spite of this deep physical difference, both regimes

FIGURE 4



A vortex filament of constant torsion with a single soliton and projections onto the x-y and x-z planes.

give rise to self-organized structure in a hot plasma. This is a persistent problem in plasma physics: What is the connection between these two classes of phenomena that are so similar qualitatively but have totally dissimilar physical origins?

Lamb's work takes an important first step in answering this question. He was able to show that the famous result shown by H. Hasimoto — that a *vortex filament can support soliton waves* — has a deep geometrical significance (see Figure 4). Specifically, he showed that the inverse scattering method for finding soliton solutions for many partial differential equations can be derived in a very natural way from the geometrical dynamics that describe vortex motion. This insight is especially crucial because until Lamb's paper the application of the inverse scattering method to the solution of a partial differential equation depended entirely on an inspired guess for the form of the transformations appropriate to the solution. Lamb was able to show that these transformations result from simple, geometric constraints of motion on a helical space curve. Note that his result came from a study of the oldest description of vortex motion in a fluid — Helmholtz's vortex filament!

CONCEPTUAL CHALLENGES

This new work and its analogues in plasma physics* raise fundamental problems — a fact *contrary* to the impression conveyed by the three articles considered here. Even though this retun to the classical outlook in physics is still in its infancy, it must be faulted for its refusal to grapple with the devastating contradictions it has uncovered in contemporary physics. The most striking of these actually have been haunting physics for some time, in various guises; but they became evident only after the development of a new, serious attack on the observable self-organizing and nonstatistical quality of the physical (and biological!) universe.

ORDER

Perhaps the most striking difficulty is that raised by *any* so-called spontaneously generated order in a physical system. Certainly our physical intuition says that physical systems obey the Second Law of Thermodynamics and evolve to states of greater disorder and greater uniformity — that the natural state of affairs is isotropic and homogeneous. There is no question that the opposite ordering processes are possible; but, goes the usual objection, they are so unlikely that they are impossible.

As Scott et al. say, in their review article on solitons: "The solitary wave was long considered a rather unimportant curiosity in the mathematical structure of nonlinear wave theory. Since it is clearly a special solution to the PDE [partial differential equation], many have assumed that somewhat special initial conditions would be required to launch it, and, therefore, that its role in relation to the initial value problem would be a minor one, at best."

The assumption has been made (and continues to be made) that self-organizing behavior is anomalous, unusual, unlikely, or, at best, a temporary fluctuation from the fundamentally linear and entropic stuff of the universe. Fluid dynamics shows experimentally that this is not the case. (To be sure, there are other, quite dramatic demonstrations of the same fact — like scientists and their research — but these have been by and large ignored.) The fact that fluid dynamics presents such a challenge is not generally recognized; conceptually, fluids demand a different global characterization of the direction of evolution of physical systems.

FIXED LAWS

The problem posed by this intuitive violation of the Second Law (formal mathematics is not the question here) is due at root to difficulties genetically characteristic of a description of a system by one, *fixed* set of laws. Fluid dynamics presents two situations where this is clear: The inability of a statistical description of turbulence arises from the

*For a review of this research see Bardwell 1976 and Bostick 1977.

presumption that a large number of degrees of freedom implies some version of Boltzmann's H-Theorem. This conjecture is based on two conditions required for the proof of an H-Theorem that do not hold for a fluid: first, the predetermination of a measurable set of normal modes that can then generate a measurable phase space; and second, a linear interaction between these modes.

If one starts with the aprioristic assumption of either a Liouvillian set of fluid elements — whose state variables and canonical coordinates determine phase space — or an atomic prescription that gets averaged into fluid elements, one cannot describe the formation of large-scale, organized motion in a satisfactory way. The fluid seems to create new elementary particles as it evolves and it is the phase space of these new entities that goes on to determine the evolution of the system. The Newtonian approach of a specifiable, perfected universe (phase space) in which a god-given set of particles interact is fundamentally incapable of producing anything but a universe that must be wound up periodically. Fluid mechanics is a sure sign that more is going on.

In a second, perhaps more basic way, the development of these recent results in fluid dynamics shows the hereditary difficulties of *any* fixed, formal description of the universe: fluid systems model, in a very basic way, the more general property of being causal but *not* deterministic. The evolution of a system modeled by the Navier-Stokes equation, for example, is notoriously difficult to describe because of several bizarre properties of the equation. Nearby solutions at one point in time can diverge arbitrarily far apart at other times, for example. This is usually taken to be evidence of entropic behavior; in fact, it occurs just at the point that we see on a macroscopic level the onset of large, coherent structure, or perhaps a phase change. In some fundamental way, the laws of the system have changed; a new phase space has been created with new dynamical entities.

This is what our formal, mathematical apparatus must deal with — but it cannot do so with *one* set of laws that must be able, before the fact, to predetermine a qualitative change in the system.

CONCEPTUAL CHALLENGES

Finally, these new lines of attack in fluid dynamics have profound implications for the foundations of physics — field theory to take one example. Mathematical physics has never dealt satisfactorily with the problem of the coexistence of continua — fields — and discrete particles. This is the basic internal inconsistency in quantum electrodynamics and lies at the root of the failure of current theories of more exotic fields. As Riemann first rigorously described it, this problem is primarily a geometric one, one of a self-generated geometry of the universe in which continua give rise to particles, which particles then mediate a change to a new geometry, which geometry then ....

Now the vortex is certainly an attractive candidate for studying this relation between continua and particles. It is the simplest, singular structure that an otherwise continuous media can support. It has a fundamental oneness about it (either the stream lines are simply connected or they are not). And, in a fluid, it has an incredibly rich, and mostly still-to-be-described behavior.

As classical physicists recognized, the most difficult question raised by vortex motion in a field theory is: What is streaming when the stream function reconnects to form a vortex? Any theory of a continuous field must deal in some way or another with the question of the so-called ether. The articles reviewed here have not, nor did the classical hydrodynamicists when they tried to make their results analogous to other continuous media. Even so, the new directions in fluid mechanics raise these field theoretic comparisons, at least implicitly. More important, they provide tools for pursuing the necessary conceptual developments. A vortex is not necessarily the underlying nonlinearlity of a field or fluid; rather, it is the signature of the underlying process of self-development and self-organization that must be understood. J.J. Thomson's famous vortex atom was based on this intuition. It is a challenge that must be taken up today.

References

Bardwell, S. 1976. "The History of the Theory and Observation of Ordered Phenomena in Magnetized Plasmas." *Fusion Energy Foundation Newsletter*: 2: 19.

Basset, A.B. 1888. *A Treatise on Hydrodynamics*. New York: Dover, 1961.

Bostick, W.H. 1977. "The Pinch Effect Revisited." *Intl. J. of Fusion Energy* 1: 1.

Brown, G. and Roshko, A. 1971. "On Density Effects and Large Structure in Turbulent Mixing Layers." *AGARD CPP*: 93; 23. See also *J. Fl. Mech.* 64: 775 (1974).

Griffin, O.M., Ramberg, S.E. 1974. "The Vortex Street Wakes of Vibrating Cylinders." *J. Fl. Mech.* 66: 553.

Hasimoto, H. 1972 "A Soliton on a Vortex Filament." *J. Fl. Mech.* 51: 477.

Kline, S.J. and Rumstadler, P.W. 1959. *J. Appl. Mech.* 26E: 166.

Lamb, H. 1932. *Hydrodynamics*. Cambridge: Cambridge Univ. Press.

Orszag, S.A. 1973. "Lecture Notes on the Statistical Theory of Turbulence." In *Les Houches Lectures* Ed. R. Balian. North-Holland Publishing Co., 1976.

Saffman, P.G. 1968. "Lectures on Homogeneous Isotropic Turbulence." In *Topics in Nonlinear Physics*. Ed. N.J. Zabusky. New York: Springer-Verlag.

Sato, H., Kuriki, K. 1961. "The Mechanics of Transition in the Wake of a Thin Flat Plate Placed Parallel to a Uniform Flow." *J. Fl. Mech.* 11: 321.

Scott, A.C., Chu, F.U., and McLaughlin, D.W. 1973. "The Soliton: A New Concept in the Physical Sciences." *Proc. IEEE* 61: 1443.

Zabusky, N. and Deem, G. 1971. "Dynamical Evolution of Two-Dimensional Unstable Shear Flows." *J. Fl. Mech.* 47: 353.